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Immersed Lagrangian Fillings of Legendrian Submanifolds via Generating Families

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by

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To my parents,
Amy and Michael Pezzimenti

Contents

Abstract

When a Legendrian submanifold admits a generating family (GF), Sabloff and Traynor proved that there is an isomorphism between the GF-cohomology groups of the Legendrian and the cohomology groups of any GF-compatible embedded Lagrangian filling. In this paper, we show that a similar isomorphism exists for immersed GF-compatible Lagrangian fillings; this imposes restrictions on the minimum number and types of double points for any such filling. We also show that from an immersed GF-cobordisms between Legendrian submanifolds, there exists a long exact sequence relating the GF-cohomology groups of the two Legendrians and cohomology groups associated to the immersed Lagrangian. In addition, we give some constructions of immersed GF-compatible fillings.

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Everything I have ever achieved is in part due to my loving parents, Amy and Michael Pezzimenti. To my mother, thank you for always pushing me to be my best, and for never, *ever* letting me give up on anything. To my father, thank you for inspiring me to pursue mathematics, and for reminding me to "enjoy the journey."

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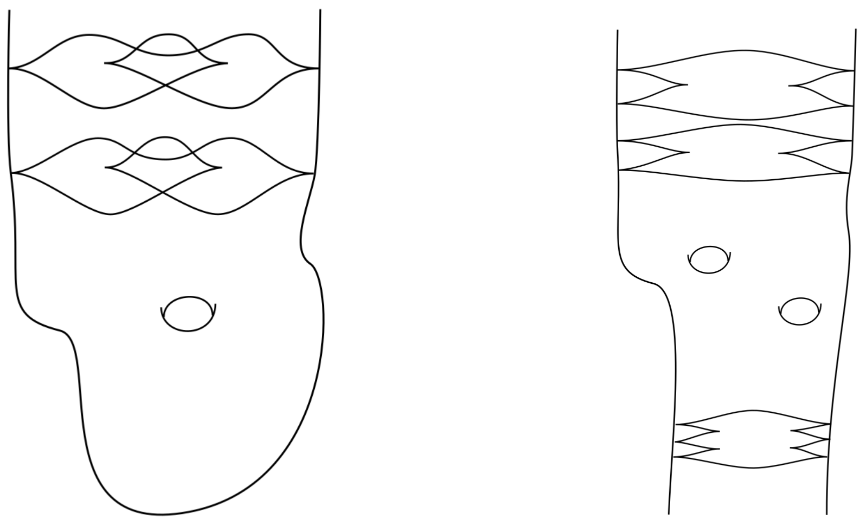
Chapter 1

Introduction

Euclidean space $\mathbb{R}^{2n}$ becomes a *symplectic* manifold when equipped with the symplectic form given by $\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Dimension n submanifolds of $(\mathbb{R}^{2n}, \omega_0)$ on which the symplectic form vanishes are known as *Lagrangian* submanifolds. Lagrangian submanifolds are extremely important in symplectic geometry. In fact, Alan Weinstein's famous "symplectic creed" asserts that "everything is a Lagrangian submanifold," meaning that important objects in symplectic geometry can be expressed in terms of Lagrangians [29].

The existence of a Lagrangian embedding of a closed manifold Σ^n into $(\mathbb{R}^{2n}, \omega_0)$ forces strong topological restrictions on Σ. For example, the torus is the only orientable surface that admits a Lagrangian embedding into $(\mathbb{R}^4, \omega_0)$. Imposing an additional exactness condition, a celebrated result of Gromov states that there is *no* exact Lagrangian embedding of a closed manifold into $(\mathbb{R}^{2n}, \omega_0)$. On the other hand, Lagrangian *immersions* are more flexible: Gromov-Lee's h-principle for Lagrangian



(a) (b)

Figure 1: (a) An embedded genus 1 Lagrangian filling of a Legendrian trefoil. (b) An embedded genus 2 Lagrangian cobordism between two Legendrian unknots.

immersions states that Σ admits a Lagrangian immersion into $(\mathbb{R}^{2n}, \omega_0)$ if and only if its complexified tangent bundle is trivial (See, for example [20], [24]). For example, this implies that every closed orientable 2-manifold admits a Lagrangian immersion into $(\mathbb{R}^4, \omega_0)$. Recently, the minimal number of double points of a Lagrangian immersion of a closed manifold has been of interest and explored in [9], [10], [11], [12], [27]. The purpose of this dissertation is to understand restrictions on the double points of an immersed orientable Lagrangian with an embedded Legendrian boundary.

Motivated by Relative Symplectic Field Theory [16], there has been a great deal of interest in Lagrangian fillings of a Legendrian submanifold and, more generally, in Lagrangian cobordisms between two Legendrian submanifolds. (For schematic pictures of these objects, see Figure 1.) Over the past ten years, significant progress has been made in understanding *embedded* Lagrangian fillings. In 2010, Chantraine [5] gave obstructions to the existence of a Lagrangian filling in terms of the Legendrian's classical invariants - the rotation and Thurston-Bennequin numbers. Further obstructions can be found in nonclassical Legendrian cohomological invariants.

In particular, given a Lagrangian filling of a Legendrian, there is an isomorphism, commonly referred to as the Seidel Isomorphism, between the topologically invariant singular relative cohomology groups of the filling and the Legendrian invariant cohomology groups of the boundary. More precisely, if f_+ is a linear-at-infinity generating family for Λ_+, and (Λ_+, f_+) admits an embedded GF-compatible filling $\mathcal{L}$ (or if ϵ_+ is an of augmentation Λ_+ induced by a filling $\mathcal{L}$), then

$$GH^k(\Lambda_+, f_+) \cong H^{k+1}(\mathcal{L}, \Lambda_+) \qquad \text{(Sabloff-Traynor [27])}, \tag{1}$$

$$LCH^{k+1}(\Lambda_+, \epsilon_+) \cong H^{k+1}(\mathcal{L}, \Lambda_+) \qquad \text{(Ekholm [8], Dimitroglou Rizell [7])}. \tag{2}$$

In (1), GH^k denotes the "relative" generating family cohomology groups of (Λ_+, f_+), defined in Definition 5 below.

Remark 1. Observe that by Poincaré duality, the isomorphism in (1) can be rewritten as $GH^k(\Lambda_+, f_+) \cong H_{n-(k+1)}(\mathcal{L})$. In particular, if $n = 2$, $GH^k(\Lambda_+, f_+) \cong H_{1-k}(\mathcal{L})$.

Furthermore, Sabloff and Traynor provide an isomorphism involving the "total" generating family cohomology groups,

$$\widetilde{GH}^k(\Lambda_+, f_+) \cong H^{k+1}(\mathcal{L}). \tag{3}$$

The total and relative generating family cohomologies are related via the following long exact sequence:

$$\cdots \to H^k(\Lambda) \to GH^k(\Lambda_+, f_+) \to \widetilde{GH}^k(\Lambda_+, f_+) \to \cdots \tag{4}$$

Defining a Poincaré polynomial with coefficients taken as either the dimensions of the Legendrian contact homology groups or the relative generating family cohomology, the isomorphism in Equation 1 has a nice interpretation for Legendrian knots and embedded, orientable Lagrangian surface fillings. Due to Sabloff's duality principle [26], the Poincaré polynomial for a Legendrian knot must be of the form

$$\Gamma_{f_+}(t) = c_n t^{-n} + ... + c_1 t^{-1} + c_0 t^{-0} + t + c_0 t^0 + c_1 t^1 + ... + c_n t^n,$$

where the coefficients $c_i \in \mathbb{Z}^+ \cup \{0\}$. We will refer to any polynomial satisfying this duality principle as a **polynomial satisfying one-dimensional duality**. There is also a duality for higher dimensional Legendrians which is described in [4]. Figure 2 lists two Legendrian $m(5_2)$ knots with the same classical invariants but different Poincaré polynomials. Applying (1) and Lefschetz duality, we can conclude the following:

- $\dim GH^0(\Lambda_+, f_+) = \dim H^1(\mathcal{L}, \Lambda_+) = \dim H_1(\mathcal{L}) = 2g$,
  where g is the genus of $\mathcal{L}$;
- $\dim GH^1(\Lambda_+, f_+) = \dim H^2(\mathcal{L}, \Lambda_+) = \dim H_0(\mathcal{L}) = 1$; and
- $GH^k(\Lambda_+, f_+)$ vanishes elsewhere.

Thus, in order for a Legendrian knot to admit an embedded GF-compatible filling of genus g, it must have polynomial $\Gamma_{f_+}(t) = t + 2g$.

A main goal of this dissertation is to extend the isomorphism in (1) and the corresponding polynomial obstruction for embedded fillings to *immersed* fillings. Although the existence of an embedded Lagrangian filling is a rather rare trait among Legendrian knots, this is not the case for immersed Lagrangians. In fact, *any* Legendrian knot with rotation number 0 has an immersed exact Lagrangian filling. (See, for example, [5].) Furthermore, Bourgeois, Sabloff and Traynor [4] show that a Legendrian with a generating family will admit an immered GF-compatible filling. In this paper, we explore what geometric restrictions exist on such fillings, including the minimum number of double points and their indices.

Using methods similar to those in [27], we produce an isomorphism between the generating family cohomology groups of the Legendrian boundary and the homology groups of a space associated to the immersed Lagrangian. To formulate this statement, let Σ denote the n-dimensional domain of the immersion and define graded groups $C(\Sigma, \{x_i\})$ with

- $\dim H_k(\Sigma)$ generators of index $k-1$ for each $k \leq n$, and
- x_i generators of index i, and x_i generators of index $-i$.

The following theorem states that if there exists a Lagrangian immersion of Σ with x_i double points of index i, then a boundary map ∂ exists, producing a chain complex $(C(\Sigma, \{x_i\}), \partial)$, whose homology groups are isomorphic to the generating family cohomology groups.

Theorem 1. *Suppose* (Λ_+, f_+) *admits an immersed GF-compatible filling* $\mathcal{L}$ *which is the immersed image of* Σ *and has* x_i *immersed double points of index* i*. If* $C(\Sigma, \{x_i\})$ *are the chain groups defined above, then there exists a boundary map* ∂ *such that*

$$GH^k(\Lambda_+, f_+) \cong H_{n-k}\left(C(\Sigma, \{x_i\}), \partial\right).$$

Remark 2. If L has no double points, the isomorphism in Theorem 1 is identical to that in (1). In particular, if $n = 2$, then $GH^k(\Lambda_+, f_+) \cong H_{2-k}\left(C(\Sigma, \{x_i\}), \partial\right) \cong H_{1-k}(\mathcal{L})$.

In Chapter 6, we will prove the existence of the chain complex $C(\Sigma, \{x_i\})$ by defining a Morse function Δ whose critical points correspond to double points of $\mathcal{L}$. This will determine the index of the double points, and the homology groups described above will be equated with the relative Morse cohomology of pairs of sublevel sets of Δ. This theorem has a useful interpretation in terms of the Poincaré polynomial for Legendrian knots.

Corollary 1. *Given a Legendrian knot* Λ_+ *with a linear-at-infinity generating family* f_+ *and Poincaré polynomial*

$$\Gamma_{f_+}(t) = c_m t^{-m} + ... + c_1 t^{-1} + c_0 t^{-0} + t + c_0 t^0 + c_1 t^1 + ... + c_m t^m, \qquad (5)$$

an immersed GF-compatible Lagrangian filling $(\mathcal{L}, F)$ *of* (Λ_+, f_+) *of genus* g *satisfies the following:*

(i) $\mathcal{L}$ *has at least* $|g - c_0| + c_1 + c_2 + \cdots + c_m$ *double points.*

(ii) $\mathcal{L}$ *has at least* c_k *immersion points of index* k*, for* $k \geq 1$.

(iii) If $g \leq c_0$*, then* $\mathcal{L}$ *has at least* $c_0 - g$ *immersion points of index 0.*

(iv) If $g > c_0$*, then* $\mathcal{L}$ *has at least* $c_1 + g - c_0$ *immersion points of index 1.*

Furthermore, for a specified Legendrian knot and generating family, the genus and number of immersed double points for all possible immersed GF-compatible fillings satisfy a modulo 2 relationship, which is described in the next theorem. This justifies the lattice configurations in Figures 3 and 4 depicting the possible immersed GF-compatible fillings of the given Legendrian knots.

Theorem 2. *Given a Legendrian knot* Λ_+ *with a linear-at-infinity generating family* f_+ *and Poincaré polynomial*

$$\Gamma_{f_+}(t) = c_m t^{-m} + ... + c_1 t^{-1} + c_0 t^{-0} + t + c_0 t^0 + c_1 t^1 + ... + c_m t^m,$$

any GF-compatible filling of (Λ_+, f_+) *of genus* g *with* p *immersed double points satisfies the following:*

$$p + g = \sum_{k=0}^{m} c_k \mod 2.$$

Remark 3. Figures 3 and 4 picture the possible immersed fillings of the Legendrian $m(5_2)$ knots in Figure 2 organized by their genus and number of double points. Theorem 2 implies that these possibilities form a lattice.

Example 1. Corollary 1 implies that any immersed GF-compatible *disk* filling of the Legendrian $m(5_2)$ knot in Figure 2a has at least one double point of index 2, and

(a) $\Gamma(t) = t^{-2} + t + t^2$ (b) $\Gamma(t) = t + 2$

Figure 2: Two Legendrian $m(5_2)$ knots with different Poincaré polynomials.

any immersed filling with genus g has an additional g double points of index 1. The possible immersed fillings of this knot are organized in the lattice in Figure 3.

Example 2. The Legendrian $m(5_2)$ knot in Figure 2b has an embedded filling of genus 1. Corollary 1 implies that any immersed GF-compatible disk filling has at least 1 double point. Any filling with genus $g \geq 2$ must have an additional g double points. The non-obstructed fillings live above the "check mark" in Figure 4.

Extending Theorem 1 to GF-compatible cobordisms between two Legendrians, we obtain a long exact sequence relating the generating family cohololology groups of (Λ_+, f_+) and (Λ_-, f_-) with homolgy groups associated to the chain complex $(C(\Sigma, \{x_i\}), \partial)$.

Theorem 3. *Suppose there exists an end-stretched GF-compatible cobordism $(\mathcal{L}, F)$ from (Λ_-, f_-) to (Λ_+, f_+), where $\mathcal{L}$ is the immersed image of Σ and has x_i double points of index i for each $i \in \{0, \cdots, m\}$. Then there exists a boundary map ∂ for $C(\Sigma, \{x_i\})$ such that the following sequence is exact:*

$$\cdots \to GH^k(\Lambda_+, f_+) \to \widetilde{GH}^k(\Lambda_-, f_-) \oplus H_{n-k}\left(C(\Sigma, \{x_i\}), \partial\right) \to \widetilde{GH}^k(\Lambda_-, f_-) \to \cdots .$$

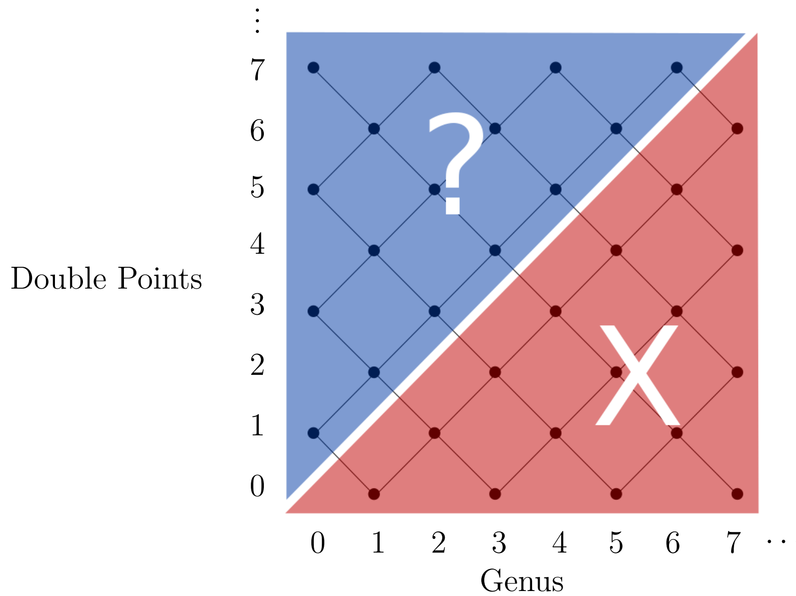


Figure 3: Possible (blue) immersed GF-compatible fillings of a Legendrian $m(5_2)$ knot with $\Gamma(t) = t^{-2} + t + t^2$.

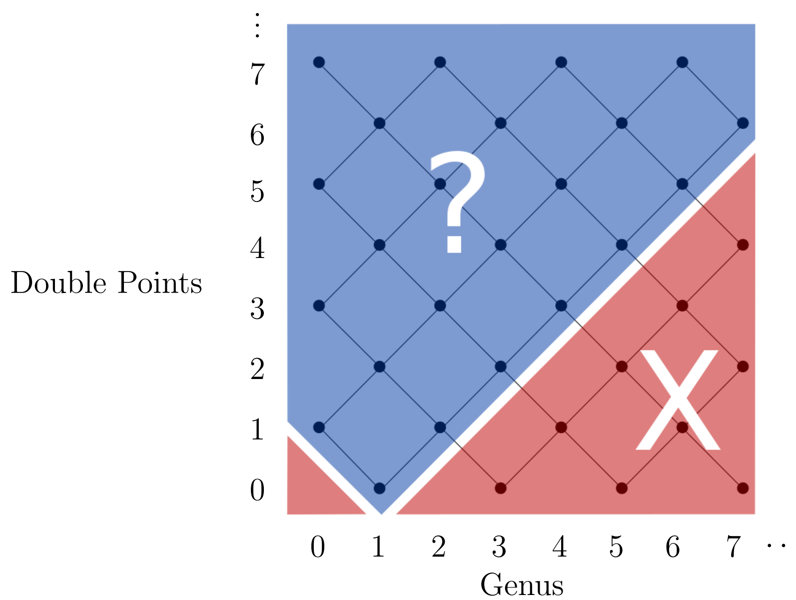
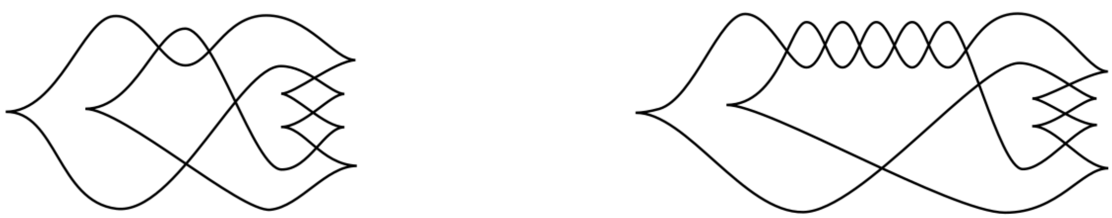


Figure 4: Possible (blue) immersed GF-compatible fillings of a Legendrian $m(5_2)$ knot with $\Gamma(t) = t + 2$.

Example 3. Let Λ_- be the $m(6_1)$ in Figure 5a and Λ_+ be the $m(10_1)$ knot in Figure 5b. Using the long exact sequence in (4), one can compute that for any generating family f_- of Λ_-, $\dim \widetilde{GH}^3(\Lambda_-, f_-) = 1$, and for any generating family f_+ of Λ_+, $\dim GH^3(\Lambda_+, f_+) = 3$. Exactness of the sequence in Theorem 3 implies that $H_{-3}\left(C(\Sigma, \{x_i\})\right) \geq 2$. This means that a genus 0 immersed end-stretched GF-compatible cobordism from (Λ_-, f_-) to (Λ_+, f_+) must have at least two double points of index 3. We will revisit this example in Chapters 6 and 7.



(a) (b)

Figure 5: (a) A Legendrian $m(6_1)$ with polynomial $t^{-3} + t + t^3$. (b) A Legendrian $m(10_1)$ with polynomial $3t^{-3} + t + 3t^3$.

With a good set of obstructions in hand, we conclude by asking which immersed GF-compatible fillings are realizable. We focus on answering this question for Legendrian knots. To do so, we construct a series of combinatorial moves that can be performed on a front diagram with a graded normal ruling. Rulings encode homological information about the generating family of the Legendrian and will be discussed more in Chapter 7. Two Legendrians whose front diagrams differ by these combinatorial moves admit an immersed GF-compatible Lagrangian cobordism. In particular, the clasping move (C) in Figure 6 will produce a Lagrangian with one immersed double point. Under certain conditions on the ruling of the Legendrian, the unclasping move (U) can be performed and will also produce a Lagrangian with one immersed double point.

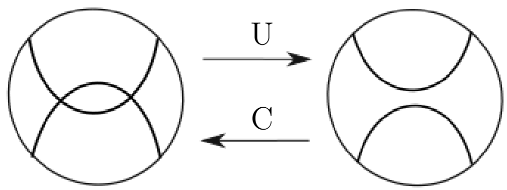


Figure 6: The (un)clasping move can be performed on a front diagram and, under certain conditions on the rulings, will result in a GF-compatible cobordism. These conditions will be outlined in Chapter 7.

Given a polynomial satisfying one-dimensional duality, namely $\Gamma(t) = c_m t^{-m} + ... + c_1 t^{-1} + c_0 t^{-0} + t + c_0 t^0 + c_1 t^1 + ... + c_m t^m$, we consider a **minimal immersed GF-compatible disk filling** to be a filling with genus 0 and c_k immersion points of index k for all $k \in \{0, ..., n\}$. Based on a construction of Melvin and Shrestha in [25], we obtain a partial converse to Corollary 1. Starting with a polynomial, we show there exists a Legendrian knot with a generating family having that polynomial and such an immersed minimal disk filling.

Theorem 4. *Given a polynomial* Γ *satisfying one-dimensional duality, there exists a Legendrian knot* Λ *with generating family* f *such that*

- $\Gamma_f = \Gamma$, *and*
- (Λ, f) *has a minimal immersed GF-compatible disk filling.*

Furthermore, from existing fillings, we construct a method of creating fillings with the same genus and additional pairs of immersion points. We can also create new fillings with higher genus at the expense of additional immersion points.

Theorem 5. *For any GF-compatible immersed filling* $(\mathcal{L}, F)$ *of* (Λ, f) *of genus* g *and any* $k \in \mathbb{Z}^+ \cup \{0\}$*,*

- *There exists another immersed GF-compatible filling* $(\mathcal{L}', F')$ *of* (Λ, f) *that has the same genus and with two additional immersion points, one of index* k *and one of index* $k+1$*;*
- *There exists another GF-compatible immersed filling* $(\mathcal{L}', F')$ *of* (Λ, f) *that has genus* $g+1$ *and one additional immersion point of index* 1*.*

We organize the data associated to a given Legendrian consisting of the obstructed, non-obstructed, and realized fillings, in an "existence lattice" as in Figure 7. Each lattice point represents an immersed GF-compatible filling of the Legendrian with a specified genus and number of double points. Theorems 1 and 4 imply that for a general polynomial Γ satisfying one-dimensional duality, there exists a Legendrian Λ with generating family f such that $\Gamma_f = \Gamma$ and such that there exists a "check mark" in the existence lattice, above which fillings of (Λ, f) are not obstructed. Theorem 4 implies that the minimal immersed GF-compatible disk filling always exists (i.e. the lowest lattice point in the green region.) The first bullet in Theorem 5 implies that the lattice points along the diagonal extending from the minimal immersed disk filling always exists, and the second bullet implies that the lattice points above this diagonal always exist.

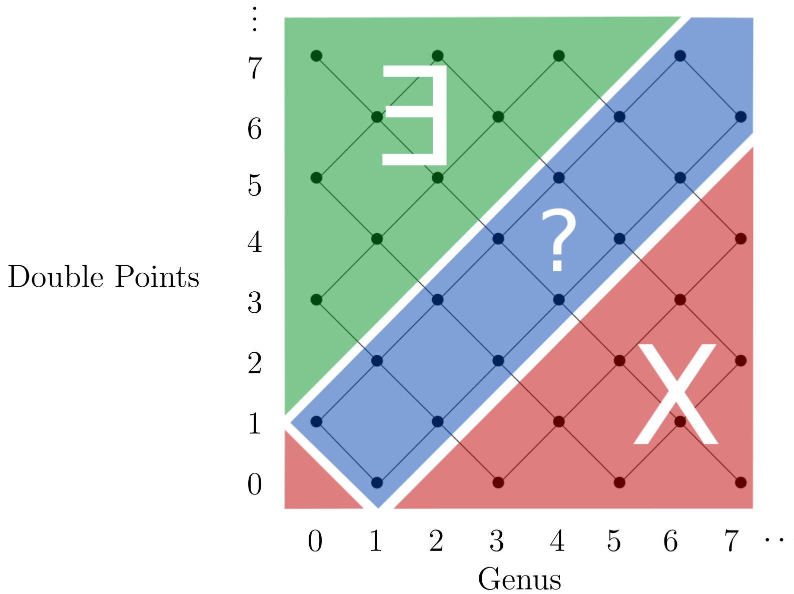


Figure 7: For a general polynomial Γ satisfying one-dimensional duality, there exists a Legendrian Λ with generating family f such that $\Gamma_f = \Gamma$ and (Λ, f) has a "check mark" (blue and green) of non-obstructed fillings. Theorems 4 and 5 imply that the lattice points in the green region always exist, perhaps with a different generating family for Λ.

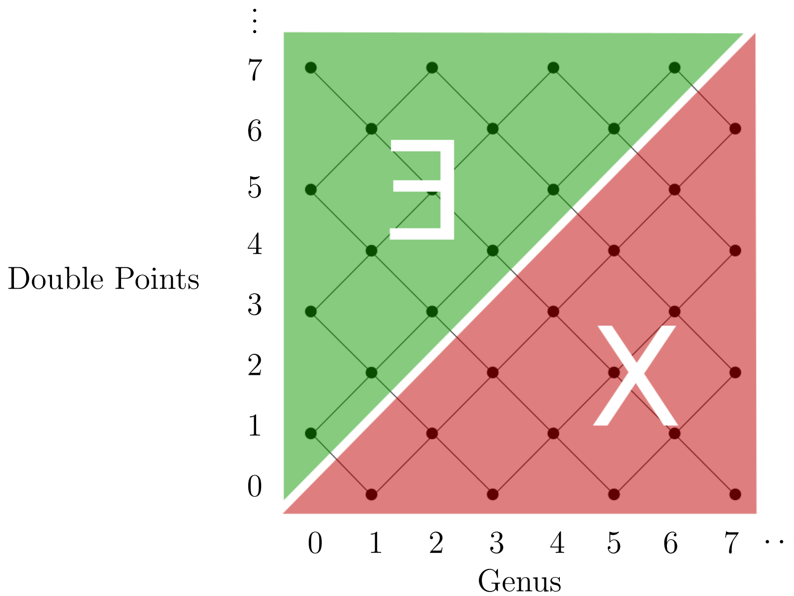


Figure 8: This lattice represents the immersed fillings that exist (green) and do not exist (red) for the Legendrian Λ, which is the $m(5_2)$ shown in Figure 2a with polynomial $t^{-2}+t+t^2$. Figure 32 in Chapter 7 shows a series of moves which proves that there exists a generating family f for Λ such that (Λ, f) has a minimal immersed disk fiiling. Theorem 5 proves the existence of the remaining lattice points.

Chapter 2

Legendrians and Lagrangians

2.1 Contact Manifolds and Legendrians

A **contact (2n+1)-manifold** is an odd dimensional manifold X together with a 1-form ξ that induces a completely non-integrable hyperplane field. The non-integrability condition is defined as follows: if ξ is locally defined as $\ker \alpha$ then $\alpha \wedge (d\alpha)^n \neq 0$. Geometrically, this means that there cannot exist an m-manifold, for $m \geq n+1$, that is everywhere tangent to the planes of a contact manifold.

The **standard contact structure** ξ_0 on $\mathbb{R}^{2n+1}$ is given by $\ker \alpha$ where $\alpha = dz - \Sigma y_i dx_i$. In $\mathbb{R}^3$, the standard contact structure $(\mathbb{R}^3, \xi_0)$ is the plane field spanned by the vectors ∂_y and $\partial_x + y\partial_z$ at each point (x, y, z). See Figure 9. The standard contact structure can more generally be placed on the 1-jet space of any smooth manifold M. Recall that the 1-jet space is given by $J^1M = T^*(M) \times \mathbb{R}$. If $(q_1, \ldots, q_n)$ are the local coordinates for M and $(q_1, p_1, \ldots, q_n, p_n)$ are the local coordinates for T^*M, then the standard contact form on J^1M is given by $\ker\alpha$ where $\alpha = dz - \Sigma p_i dq_i$.

An important submanifold of a contact manifold that will be of interest for this dissertation is a *Legendrian* submanifold. Given a $2n{+}1$-dimensional contact manifold (J^1M, ξ), an n-dimensional submanifold $\Lambda \subset M$ is **Legendrian** if its tangent space satisfies $T_p\Lambda \subset \xi$ for all $p \in \Lambda$.

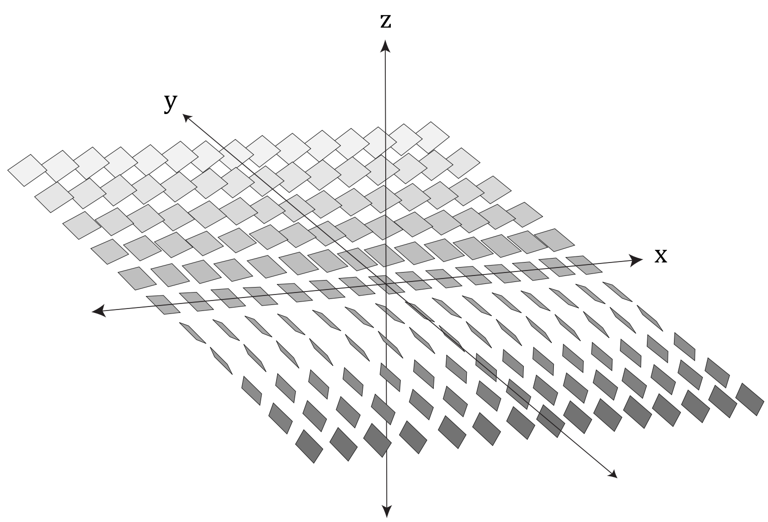


Figure 9: Standard Contact Structure $(\mathbb{R}^3, \xi_0)$

From a contact manifold M, we define its **Reeb vector field** to be the vector field $R_\alpha : M \to TM$ such that

$$d\alpha(R_\alpha, \cdot) = 0, \quad \text{and} \quad \alpha(R_\alpha) = 1.$$

For the standard contact structure on $J^1\mathbb{R}$, this is just given by ∂_z. A **Reeb chord** of a Legendrian is a trajectory of the Reeb vector field that intersects the Legendrian at two distinct points. For a Legendrian knot in the standard contact structure, these are vertical trajectories that hit the Legendrian at both ends (points with the same x-coordinate and slope in the xz-projection).

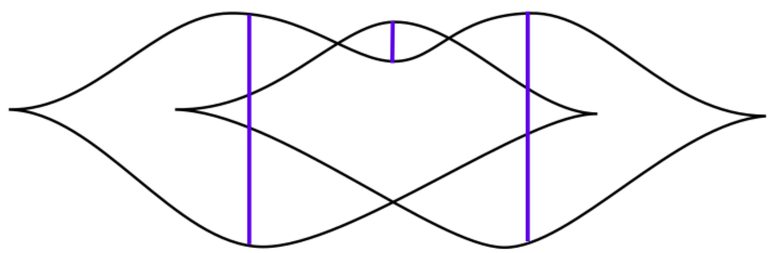


Figure 10: The front projection of a Legendrian trefoil knot with three of its Reeb chords shown.

2.2 Legendrian Knots and Invariant Groups

As with smooth knots, we generally work with projections of Legendrian knots. We call the xz-projection of the knot the **front projection** and the xy-projection the **Lagrangian projection**. Since the contact planes are never vertical, a Legendrian knot can have no vertical tangencies in its front projection. Instead, it has "cusps". See Figure 10. Notice that this diagram does not specify which strand is the over-strand in each crossing. Since we always view the y-axis as pointing into the page, and the contact planes increase slope as they move in the positive y direction, the strand with the lesser slope will always be the overstrand.

There are several invariants that are useful in classifying Legendrian knots. Since every Legendrian knot is also a smooth knot, the underlying smooth knot type is an invariant of Legendrian knots. Two other important invariants, known as the **classical invariants**, are the Thurston-Bennequin and rotation numbers. For Legendrian knots, these invariants can be calculated easily from their front projections.

The **Thurston-Bennequin number**, $tb(\Lambda)$, is the linking number between Λ and a push-off Λ' of Λ in the positive z direction and can be calculated from the front projection as follows:

$$tb(\Lambda) = w(\Lambda) - \frac{1}{2}C$$

where the *writhe* $w(\Lambda)$ is the number of positive crossings minus the number of negative crossings, and C is the number of cusps. The **rotation number** $r(\Lambda)$ is defined for an oriented Legendrian knot and is given by:

$$r(\Lambda) = \frac{1}{2}(D - U)$$

where D is the number of cusps oriented downward and U is the number of cusps oriented upwards. Figure 11 shows some calculations of tb and r.

These classical invariants can be used to distinguish between Legendrian knots that would otherwise be isotopic as smooth knots. For example, each point in the "mountain range" in Figure 11 represents a distinct Legendrian unknot with a specified set of classical invariants.

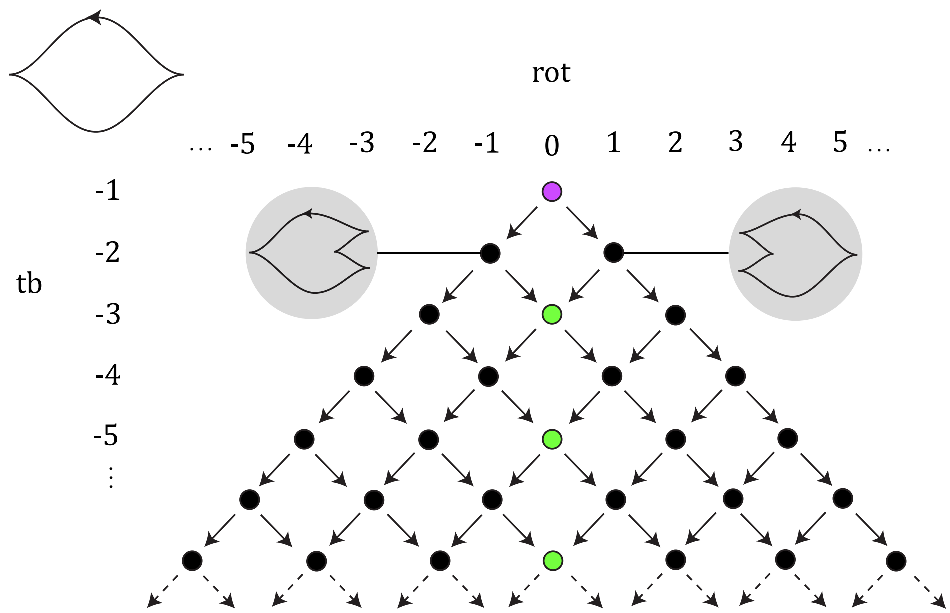


Figure 11: This mountain range classifies all Legendrian unknots based on their rotation and Thurston Bennequin numbers.

2.3 Symplectic Manifolds and Lagrangians

For a smooth $2n$-dimensional manifold B, recall that a differential p-form ω on B is **closed** if $d\omega = 0$. We say that ω is **non-degenerate** if for all $p \in B$ and for all non-zero $\vec{v} \in T_pB$, there exists $\vec{w} \in T_pB$ such that $\omega(\vec{v}, \vec{w}) \neq 0$. A **symplectic manifold** is a pair (B, ω) where ω is a closed, nondegenerate, differential 2-form. A symplectic manifold is **exact** if $\omega = d\lambda$ for some λ. We call λ a **primitive** of ω.

There are two symplectic spaces we will work in for the purposes of this paper. The **standard symplectic form** on $\mathbb{R}^{2n}$ is given by

$$\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$$

This form is exact with preferred primitive

$$\lambda = x_1 dy_1 + \ldots + x_n dy_n.$$

The standard symplectic form can be more generally placed on the cotangent bundle of any smooth manifold B. If $(q_1, p_1, \ldots, q_n, p_n)$ are the local coordinates for T^*B, then the standard symplectic form is given by $dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n$.

From a contact manifold $(J^1M, \ker \alpha)$, we can obtain the symplectic manifold, $(\mathbb{R} \times J^1M, e^s\alpha)$. We call this the **symplectization of a contact manifold**. These two spaces are equivalent via the following symplectomorphism:

$$\theta : \mathbb{R} \times J^1M \to T^*(\mathbb{R}_+ \times M)$$

$$(s, x, y, z) \mapsto (e^s, x, z, e^s y).$$

A submanifold L of a symplectic manifold (B^{2n}, ω) is **isotropic** if $\omega|_L = 0$, that is, if for all $p \in L$ and for all $\vec{v}, \vec{w} \in T_pL$, $\omega(\vec{v}, \vec{w}) = 0$. If an isotropic submanifold is also n-dimensional, we call it a **Lagrangian**. A Lagrangian submanifold L is **exact** if $\lambda|_L = df$, for some function f.

2.4 Lagrangian Cobordisms

Lagrangians are the "even-dimensional analog" of Legendrians. There is an interplay between Legendrians and Lagrangians through embedded cobordism in that an embedded Lagrangian submanifold can have Legendrians as boundary components. To define this more rigorously, we first define the **cylinder** over a Legendrian Λ to be the Lagrangian submanifold $\Lambda \times \mathbb{R} \subset J^1M \times \mathbb{R}$. A Lagrangian cobordism, L, from a Legendrian Λ_- to a Legendrian Λ_+ is a Lagrangian submanifold that is cylindrical over Λ_- and Λ_+ at its ends. We can also consider *immersed* Lagrangian cobordisms between *embedded* Lagrangian submanifolds, as in the following.

Definition 1. An **immersed Lagrangian cobordism** $L \subset \mathbb{R} \times J^1M$ from Λ_- to Λ_+ is an immersed Lagrangian submanifold such that for some $s_-, s_+ \in \mathbb{R}^+$, we have

- $L \cap (\{t\} \times J^1M) = \{t\} \times \Lambda_-$ whenever $t < s_-$ and
- $L \cap (\{t\} \times J^1M) = \{t\} \times \Lambda_+$ whenever $t > s_+$.

If L is exact, we say L is an **exact Lagrangian cobordism**. If $\Lambda_- = \emptyset$, we say L is a **filling**.

In the next section, we describe a method of defining Legendrians and Lagrangians in terms of generating families. In particular, we explain how a Lagrangian cobordism can be given a generating family that is "compatible" with its Legendrian ends.

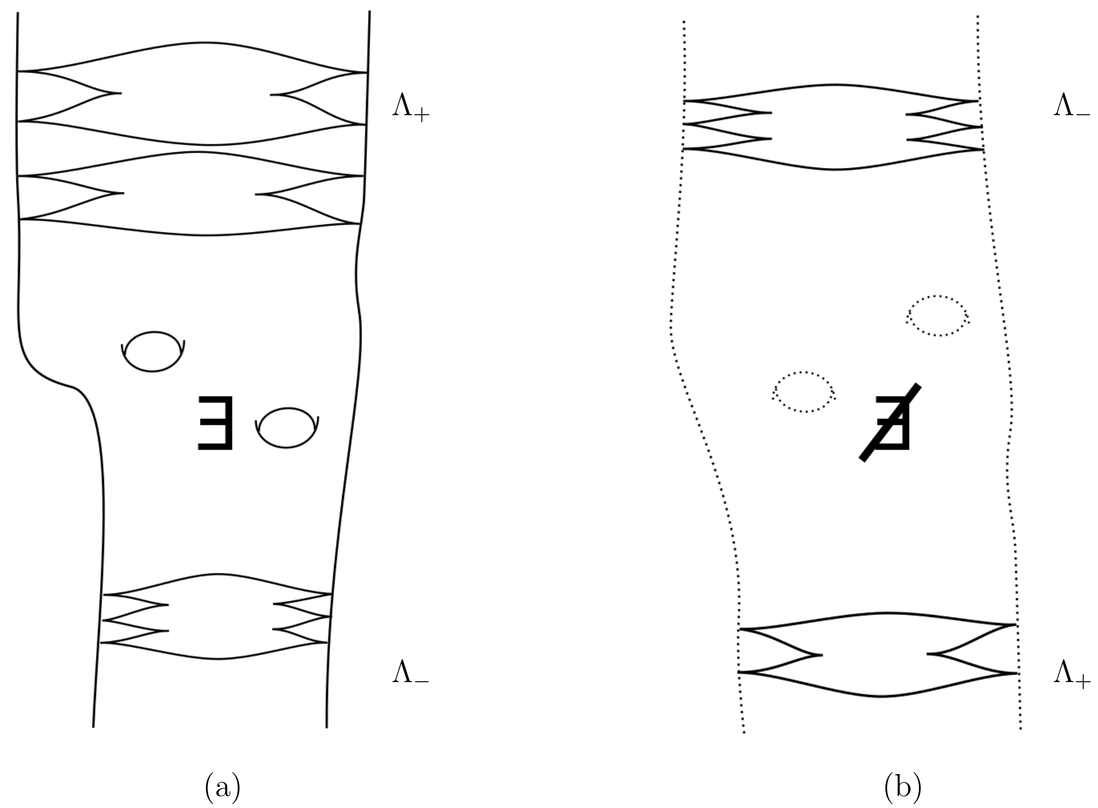


Figure 12: (a) There exists an exact Lagrangian cobordism from Λ_- to Λ_+. (b) There does not exist an exact Lagrangian cobordism from Λ_+ to Λ_-.

Chapter 3

Generating Families

3.1 Generating Families for Legendrians and Lagrangians

While many techniques have been used to study Legendrian submanifolds, we will be primarily working with *generating families*, a way of describing Legendrians through funtions. Given a smooth manifold M and a function $f : M \to \mathbb{R}$, its 1-jet,

$$j^1 f = \{(x, Df(x), f(x))\}$$

defines a Legendrian in the 1-jet space, $J^1 M = T^*(M) \times \mathbb{R}$, where Df denotes all the Jacobian of all partial derivatives. For Legendrians that cannot be described in this way, we extend to a generating *family* of functions.

Suppose $f : M \times \mathbb{R}^N \to \mathbb{R}$ is a smooth function and that 0 is a regular value of

$\frac{\partial f}{\partial \eta}$. We will call the submanifold $\Sigma_f = \left(\frac{\partial f}{\partial \eta}\right)^{-1}(0)$ the **fiber critical set** and define on it a map

$$j_f : \Sigma_f \to J^1 M, \text{ where}$$

$$j_f(x, \eta) = \left(x, \frac{\partial f}{\partial x}(x, \eta), f(x, \eta)\right).$$

The image of j_f is a potentially immersed Legendrian submanifold. We say that a Legendrian, Λ, is generated by f, or f is the **generating family** for Λ, if $\Lambda = j_f(\Sigma_f)$.

Whereas Legendrians can arise from the 1-jet of a function, $F : B \to \mathbb{R}$, Lagrangians can arise from the graph of its derivative,

$$\Gamma_{DF} = \{(x, DF(x))\} \subset T^*B.$$

We can similarly extend this idea to define generating families for Lagrangians. For a smooth map, $F : B \times \mathbb{R}^N \to \mathbb{R}$ such that 0 is a regular value of $\frac{\partial F}{\partial \eta}$, we define

$$\partial_F : \Sigma_F \to T^*B,$$

$$\partial_F(x, \eta) = \left(x, \frac{\partial F}{\partial x}(x, \eta)\right).$$

The image of ∂_F is a (potentially immersed) Lagrangian submanifold. We say that a Lagrangian, $\mathcal{L}$, is generated by F, or F is the **generating family** for $\mathcal{L}$, if $\mathcal{L}$ is equal to the image of ∂_F. We always assume F is generic so that immersed points are always double points.

For the remainder of this paper, we will be working with Legendrians and Lagrangians that have generating families. Thus, our Legendrians will be submanifolds of the 1-jet of a smooth manifold, J^1M and Lagendrians will be submanifolds of the cotangent bundle of a smooth manifold, T^*B. As mentioned above, we would like to be able to use Morse-theoretic techniques on these functions. Since J^1M is non-compact, it will often be necessary to impose the following linearity condition on our generating families. A function $f : M \times \mathbb{R}^N \to \mathbb{R}$ is **linear-at-infinity** if it can be written as the sum

$$f(x, \eta) = f_c(x, \eta) + A(\eta)$$

of a compactly supported function f_c and a non-zero linear function A.

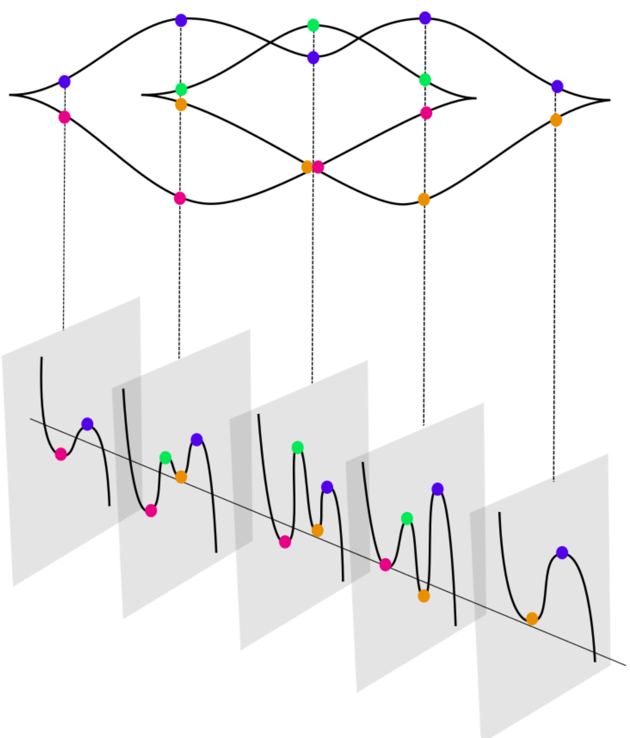


Figure 13: A generating family for a Legendrian trefoil.

3.2 GF-compatible Lagrangian Cobordisms

In order to describe a Lagrangian cobordism with generating families, we will apply the syplectomorphism:

$$\theta : \mathbb{R} \times J^1 M \to T^* (\mathbb{R}_+ \times M)$$

$$(s, x, y, z) \mapsto (e^s, x, z, e^s y)$$

and consider $\mathcal{L} = \theta(L)$ to be the immersed Lagrangian cobordism living in $T^*(\mathbb{R}_+ \times M)$. Given Legendrians $\Lambda_\pm$ generated by $f_\pm : M^m \times \mathbb{R}^N \to \mathbb{R}$, we would like to define a Lagrangian cobordism L that has a generating family "compatible" with $f_\pm$.

Definition 2. Given generating families $f_\pm : M \times \mathbb{R}^N \to \mathbb{R}$ of Legendrians $\Lambda_\pm$, we say

$$F : (\mathbb{R}_+ \times M) \times \mathbb{R}^N \to \mathbb{R}$$

extends $f_\pm$ if F generates a (potentially immersed) Lagrangian $\mathcal{L}$ and for some values $t_- < t_+$, we have

$$F(t, x, \eta) = \begin{cases} t f_-(x, \eta), & t \leq t_-, \\ t f_+(x, \eta), & t \geq t_+. \end{cases}$$

To aid in future calculations, we will require that F, f_-, and f_+ satisfy the following conditions:

Definition 3. A function $F : (\mathbb{R}_+ \times M) \times \mathbb{R}^N \to \mathbb{R}$ is **slicewise-linear-at-infinity** if for all $t \in \mathbb{R}_+$, there exists a compactly supported function $F_t^c : M \times \mathbb{R}^N \to \mathbb{R}$ and a non-zero linear function $A_t : \mathbb{R}^N \to \mathbb{R}$ so that $F(t, x, \eta) = F_t^c(x, \eta) + A_t(\eta)$. A triple of functions (F, f_-, f_+) satisfying the GF-compatibility condition is called **tame** if F is slicewise-linear-at-infinity and $f_\pm$ are linear-at-infinity.

Given that F, f_-, and f_+ satisfy these tameness conditions, the lemma below follows from Definition 2.

Lemma 1. *If F extends $f_\pm$, then F generates a (potentially immersed) exact Lagrangian cobordism from Λ_- to Λ_+.*

Definition 4. A cobordism of the type described in Lemma 1 is called an **(immersed) GF-compatible cobordism**. If $\Lambda_- = \emptyset$, we call it an **(immersed) GF-compatible filling**.

Remark 4. All GF-compatible cobordisms are necessarily exact.

3.3 Generating Family Cohomology

For Legendrians with generating families, we can define a cohomology group that captures information about its Reeb chords. Given a contact form α, recall that the Reeb vector field R satisfies $R \in \ker d\alpha, \alpha(R) = 1$, and a Reeb chord is an integral curve of R with positive length and with both endpoints on the Lagrangian. To each generating family, we shall define an associated **difference function**, $\delta : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by

$$\delta(x, \eta, \tilde{\eta}) = f(t, x, \tilde{\eta}) - f(t, x, \eta). \tag{6}$$

The critical points of δ with non-zero critical value are in 2-1-correspondence with the Reeb chords of Λ.

Proposition 6 (Proposition 3.1 in [27])**.** *For each Reeb chord γ with length $l(\gamma)$, there are two critical points $(x, \eta, \tilde{\eta})$ and $(x, \tilde{\eta}, \eta)$ of δ with critical values $\pm l(\gamma)$. All other critical points of δ lie in the non-degenerate critical submanifold,*

$$\{(x, \eta, \eta) : (x, \eta) \in \Sigma_f\}$$

and have critical value 0*.*

Because the critical points and values of this difference function carry strong geometric meaning, it is natural to apply Morse theoretic techniques to sublevel sets of δ,

$$\delta^a = \{(x, \eta, \tilde{\eta}) : \delta(x, \eta, \tilde{\eta}) \leq a\}.$$

Let $\underline{l}$ denote the smallest Reeb height of Λ, and let $\bar{l}$ denote the largest. We may then define the following cohomology groups of a Legendrian with generating family, (Λ, f).

Definition 5. Let positive constants ω and ϵ be chosen so that

$$0 < \epsilon < \underline{l} < \bar{l} < \omega. \tag{7}$$

The **relative generating family cohomology** of f is given by

$$GH^k(f) = H^{k+N+1}(\delta^\omega, \delta^\epsilon).$$

The **total generating family cohomology** of f is given by

$$\widetilde{GH}^k(f) = H^{k+N+1}(\delta^\omega, \delta^{-\epsilon}).$$

Remark 5. (a) Here, we take $H^*(\delta^\omega, \delta^\epsilon)$ to be the dual to the singular homology of the pair of sublevel sets with coefficients are taken over a field.

(b) Generating family *homology* groups can be defined using an analogous definition.

(c) The degree shift of N is chosen to account for stabilization in the generating family by a quadratic and the degree shift of $+1$ is chosen so that the generating family homology groups agree with the linearized contact homology groups.

It has been shown that for a linear-at-infinity generating family, $GH^k(f)$ does not depend on choice of ω and ϵ (see, for example [27]). By Poposition 6, ω and ϵ are chosen such that all positive critical values of δ lie in $[\epsilon, \omega]$. For any other ω' and ϵ' satisfying (7), a Morse-theoretic argument can be used to show that the pair $(\delta^\omega, \delta^\epsilon)$ is a deformation retract of $\left(\delta^{\omega'}, \delta^{\epsilon'}\right)$.

It should also be noted that these cohomology groups are associated to a particular choice of generating family for the Legendrian. The generating family cohomology groups of equivalent generating families will remain equal. However, if the Legendrian is redefined by a non-equivalent generating family, the cohomology groups may change. We get an invariant by taking the set of *all* generating family cohomology groups taken over *all* generating families of a Legendrian. That is, we have the following:

Proposition 7. *(Traynor [28]) For a Legendrian $\Lambda \in J^1M$, the set*

$$\left\{GH^k(f) : f \text{ generates } \Lambda\right\}$$

is invariant under Legendrian isotopy.

Consequently, we can define the following polynomial invariant.

Definition 6. For a Legendrian $\Lambda \in J^1M$, the set

$$\left\{\Gamma_f(t) = \sum_k \dim(GH^k(f))t^k : f \text{ generates } \Lambda\right\}$$

is invariant under Legendrian isotopy. Each $\Gamma_f(t)$ is a **Poincaré polynomial of** Λ.

Chapter 4

Wrapped Generating Family Cohomology

With the notion of a GF-compatible Lagrangian cobordism between Legendrian submanifolds in mind, we seek a homology theory that detects information both about the topology of the domain of the Lagrangian immersion, the double points in the image, *and* the Reeb chords of the Legendrian ends. As described in [27], we invoke the notion of wrapped Floer homology (see, for example, [1], [2], and [19]) by building a chain complex generated by intersections of the Lagrangian with its image under an appropriately defined Hamiltonian shift.

4.1 Pre-Sheared Difference Function

Before defining the Hamiltonian function, let us first analyze the difference function associated to a generating family F for a Lagrangian $\mathcal{L} \subset T^*(\mathbb{R}_+ \times M)$. We will call

$\Delta_0 : \mathbb{R}_+ \times M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ the **difference function** of F and define it by

$$\Delta_0(t, x, \eta, \tilde{\eta}) = F(t, x, \tilde{\eta}) - F(t, x, \eta). \tag{8}$$

As with δ, the critical points and values of Δ_0 capture geometric information. When $t \in [t_-, t_+]$, Δ_0 has an embedded, non-degenerate critical submanifold $\mathcal{C}$ with critical value 0, as well as pairs of critical points of opposite critical value for each immersed double points of the Lagrangian:

Theorem 8. *The set of critical points of Δ_0 with $t \in [t_-, t_+]$ is equal to the set $\mathcal{D} \cup \mathcal{C}$, where*

$$\mathcal{D} = \{(t, x, \eta, \tilde{\eta}) : (t, x, \eta) \neq (t, x, \tilde{\eta}) \in \Sigma_F \text{ and } \partial_F(t, x, \eta) = \partial_F(t, x, \tilde{\eta})\},$$

and

$$\mathcal{C} = \{(t, x, \eta, \eta) : (t, x, \eta) \in \Sigma_F \text{ and } t \in [t_-, t_+]\}.$$

The non-degenerate critical submanifold, $\mathcal{C}$ has critical value 0, and is diffeomorphic to the fiber critical set, Σ_F. For each immersed double point in $\mathcal{L}$, there is a pair of critical points in $\mathcal{D}$ whose critical values are negatives of one another. This value is equal to $\int_\gamma \lambda$ where $\gamma = \partial_F \circ h$ is a closed curve on $\mathcal{L}$ and h is a path from (t, x, η) to $(t, x, \tilde{\eta})$.

Remark 6. Points $(t, x, \eta, \tilde{\eta})$ and $(t, x, \tilde{\eta}, \eta)$ of $\mathcal{D}$ correspond to the same immersed double point of $\mathcal{L}$.

Proof. For any $t \in [t_-, t_+]$, the vanishing of all partial derivatives of a critical point

$(t, x, \eta, \tilde{\eta})$ of Δ_0 imply the following:

$$0 = -\frac{\partial F}{\partial \eta}(t, x, \eta) = \frac{\partial F}{\partial \tilde{\eta}}(t, x, \tilde{\eta}), \tag{9}$$

and

$$0 = \frac{\partial F}{\partial t}(t, x, \tilde{\eta}) - \frac{\partial F}{\partial t}(t, x, \eta) = \frac{\partial F}{\partial x}(t, x, \tilde{\eta}) - \frac{\partial F}{\partial x}(t, x, \eta). \tag{10}$$

Equation (9) implies that $(t, x, \eta), (t, x, \tilde{\eta}) \in \Sigma_F$, and Equation (10) implies that $\partial_F(t, x, \eta) = \partial_F(t, x, \tilde{\eta})$. If $\tilde{\eta} = \eta$, then $(t, x, \eta, \tilde{\eta})$ is an element of $\mathcal{C}$, and if $\tilde{\eta} \neq \eta$, then $(t, x, \eta, \tilde{\eta})$ is an element of $\mathcal{D}$.

It remains to compute the critical values. If (t, x, η, η) is a point of $\mathcal{C}$, then its critical value is

$$\Delta_0(t, x, \eta, \eta) = F(t, x, \eta) - F(t, x, \eta) = 0.$$

The calculation of the critical value for a point in $\mathcal{D}$ is detailed in Lemma 2 below and will complete the proof.

□

Lemma 2. *If $(t, x, \eta, \tilde{\eta}) \in \mathcal{D}$ is a critical point of Δ, then the critical value,*

$$\Delta(t, x, \eta, \tilde{\eta}) = \int_\gamma \lambda,$$

where $\gamma = \partial_F \circ h$ is a closed curve on $\mathcal{L}$ and h is a path from (t, x, η) to $(t, x, \tilde{\eta})$.

Proof. Fix a path $h : [0, 1] \to \Sigma_F$ from $(t, x, \tilde{\eta})$ to (t, x, η). Then

$$\gamma = \partial_F \circ h : [0, 1] \to T^*(\mathbb{R}_+ \times M)$$

is a closed loop on $\mathcal{L}$ at $\partial_F(t,x,\tilde{\eta}) = \partial_F(t,x,\eta)$. We then have:

$$\int_{\gamma[0,1]} \lambda = \int_{\partial_F \circ h[0,1]} \lambda = \int_{h[0,1]} \partial_F^* \lambda. \tag{11}$$

We shall show that $\partial_F^* \lambda = d\bar{F}$, where $\bar{F} : \Sigma_F \to \mathbb{R}$, is the restriction of F to the fiber critical set, Σ_F.

Take $(q_1, q_2, q_3, p_1, p_2, p_3)$ to be coordinates of $T^*((\mathbb{R}_+ \times M) \times \mathbb{R}^N)$ and (q_1, q_2, p_1, p_2) to be the coordinates for $T^*(\mathbb{R}_+ \times M)$. Then the primitive of the symplectic form $\bar{\omega}$ associated to $T^*((\mathbb{R}_+ \times M) \times \mathbb{R}^N)$ is

$$\bar{\lambda} = p_1 dq_1 + p_2 dq_2 + p_3 dq_3,$$

and the primitive of the symplectic form ω associated to $T^*(\mathbb{R}_+ \times M)$ is

$$\lambda = p_1 dq_1 + p_2 dq_2.$$

With this in mind, we define the following maps:

$$\bar{\partial} : (\mathbb{R}_+ \times M) \times \mathbb{R}^N \to T^*((\mathbb{R}_+ \times M) \times \mathbb{R}^N)$$
$$(t,x,\eta) \mapsto \left(t, x, \eta, \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial \eta}\right),$$

and

$$\partial : \Sigma_F \to T^*((\mathbb{R}_+ \times M) \times \mathbb{R}^N) \cap \{p_3 = 0\}$$
$$(t,x,\eta) \mapsto \left(t, x, \eta, \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, 0\right).$$

Letting $A = T^*((\mathbb{R}_+ \times M) \times \mathbb{R}^N) \cap \{p_3 = 0\}$, we see that

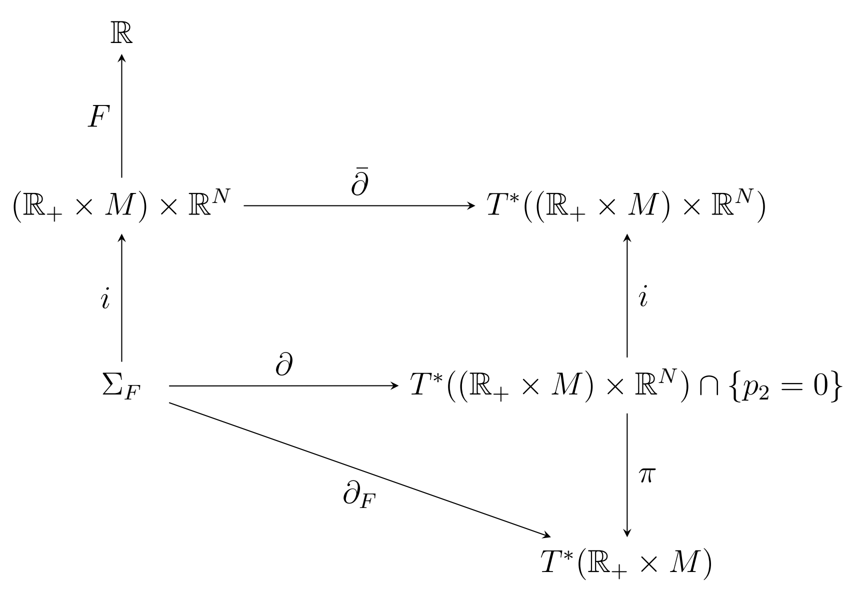
$$A^\perp = \{v : \omega(v, w) = 0, \forall w \in T(A)\} = q_3,$$

and thus, we can define

$$\pi : T^*((\mathbb{R}_+ \times M) \times \mathbb{R}^N) \cap \{p_3 = 0\} \to T^*(\mathbb{R}_+ \times M)$$

$$(q_1, q_2, q_3, p_1, p_2, 0) \mapsto (q_1, q_2, p_1, p_2)$$

to be the characteristic foliation along q_3. Letting $i : A \hookrightarrow T^*((\mathbb{R}_+ \times M) \times \mathbb{R}^N)$ and $\bar{i} : \Sigma_F \hookrightarrow (\mathbb{R}_+ \times M) \times \mathbb{R}^N$ be the inclusion maps, we get the following commutative diagram:



Therefore, the following equalities hold:

$$\begin{aligned}\pi^*(\lambda) &= \pi^*(p_1 dq_1 + p_2 dq_2)\\ &= p_1 dq_1 + p_2 dq_2\\ &= i^*(p_1 dq_1 + p_2 dq_2 + p_3 dq_3)\\ &= i^*\bar{\lambda}, \end{aligned} \tag{12}$$

and

$$\begin{aligned}\bar{\partial}^*(\bar{\lambda}) &= \bar{\partial}^*(p_1 dq_1 + p_2 dq_2 + p_3 dq_3)\\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial \eta}d\eta\\ &= dF. \end{aligned} \tag{13}$$

From Equations (12) and (13), and by commutativity of the diagram, we arrive at the following:

$$\partial_F^*\lambda = \partial^*\pi^*\lambda = \partial^*(i^*\bar{\lambda}) = (\bar{\partial}\circ i)^*\lambda = i^*(\bar{\partial}^*(\bar{\lambda})) = i^* dF = d\bar{F}. \tag{14}$$

This gives us our desired equality, and we can conclude:

$$\int_\gamma \lambda = \int_h \partial_F^*\lambda = \int_h d\bar{F} = \int_{\partial h} \bar{F} = \bar{F}(h(1)) - \bar{F}(h(0)) = \Delta(t, x, \eta, \tilde{\eta}).$$

□

4.2 Top-stretched Fillings and Stretched Cobordisms

Theorem 8 shows that the difference function captures the topology of $\mathcal{C}$, which is the domain of the immersion, and $\mathcal{M}$, the number of immersed double points. However, after some slight modifications to the difference function, it will also capture the Reeb chords of the Legendrian ends. In particular, we will tweak the difference function to a *sheared* difference function by defining a Hamiltonian shearing function.

This idea is described by Sabloff and Traynor in [27] in order to define the wrapped generating family cohomology groups for *embedded* Lagrangians with cylindrical ends. The shearing function allows us to associate Reeb chords of the Legendrian with intersection points of the Lagrangian and its image under the appropriate Hamiltonian function. The schematic picture in Figure 14 identifies the critical values of Δ for an embedded GF-comaptible filling.

When considering immersed fillings, we also have the additional critical points of Δ arising from the immersed double points, namely those in $\mathcal{D}$. In Figure 15, these pairs of critical values are all shown to lie in the region $[-\mu, \mu]$. However, a priori, this need not be the case. This condition is convenient for later analysis arguments in order to capture all critical values of Δ when taking cohomology. We will show that by performing a Legendrian isotopy, we can ensure this occurs.

To that end and with Lemma 8 in mind, let

$$\mu_{\mathcal{D}}(F) = \max_{\alpha \in \mathcal{D}} \left\{ |\Delta_0(\alpha)| \right\}, \tag{15}$$

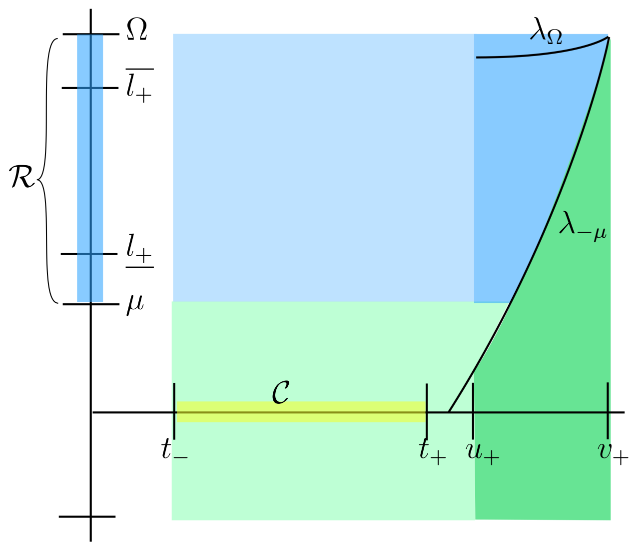


Figure 14: A schematic picture of the critical points and values of Δ for an embedded filling. The values Ω and μ are chosen so that critical values of the set of critical points of $\Delta|_{[u_+,v_+]}$, which we call $\mathcal{R}$, lie within $[\mu, \Omega]$. We will see in a future lemma that these correspond to the positive critical values of δ. Notice that $\left(\Delta^{\Omega}_{[u_+,v_+]}, \Delta^{-\mu}_{[u_+,v_+]}\right)$ can be identified with the cone, $C(\delta^\omega, \delta^\epsilon)$. This will be shown more rigorously in the proof of Lemma 9 and is the key to proving Theorem 11. Furthermore, there is a critical submanifold with critical value 0 that can be identified with Σ_F (or $\mathcal{C}$), and captures the topology of the filling.

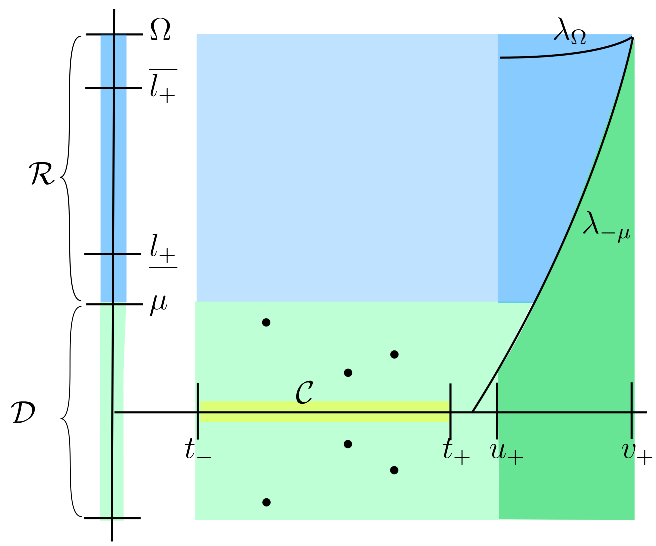


Figure 15: A schematic picture of the critical points and values of Δ for an immersed filling. Notice that this diagram is essentially equivalent to that in Figure 14 with the additional set $\mathcal{D}$ of pairs of critical points within the region $[t_-, t_+]$ having opposite critical values. Each of these pairs of critical points correspond to immersed double points in the Lagrangian.

the largest critical value of Δ_0 with $t \in [t_-, t_+]$. We then define the following:

Definition 7. An immersed GF-compatible Lagrangian cobordism $(\mathcal{L}, F)$ is called **top-stretched** if

$$4\mu_{\mathcal{D}}(F) < t_+ \underline{l_+}.$$

An immersed GF-compatible Lagrangian cobordism $(\mathcal{L}, F)$ is **end-stretched** if

$$4\mu_{\mathcal{D}}(F) < \max\left\{t_+ \underline{l_+}, t_- \underline{l_-}\right\}.$$

For ease of calculation in future lemmas, we will restrict our focus to top-stretched fillings and stretched cobordisms. In the proposition below, we show that any immersed GF-compatible Lagrangian filling (cobordism) can be stretched to a top-stretched (stretched) one.

Proposition 9. *Let $(\mathcal{L}, F)$ be an immersed GF-compatible cobordism from (Λ_-, f_-) to (Λ_+, f_+). Suppose L is cylindrical outside the region $[t_-, t_+]$. Then there exists a value $\widetilde{t_+} > t_+$ and a Legendrian $\widetilde{\Lambda_+}$ with generating family $\widetilde{f_+}$ such that:*

- *$\widetilde{\Lambda_+}$ is Legendrian isotopic to Λ_+, and differs only by a z-direction stretch;*
- *$\widetilde{f_+}$ is homotopic to f_+; and*
- *there exists a top-stretched cobordism $\left(\widetilde{\mathcal{L}}, \widetilde{F}\right)$ from (Λ_-, f_-) to $\left(\widetilde{\Lambda_+}, \widetilde{f_+}\right)$ that is cylindrical outside the region $[t_-, \widetilde{t_+}]$.*

Furthermore, $\widetilde{\mathcal{L}}$ is homeomorphic to $\mathcal{L}$ and $\widetilde{\mathcal{L}}|_{[t_+, \widetilde{t_+}]}$ is a concordance.

Proof. Our strategy is to remove the top cylindrical portion of $(\mathcal{L}, F)$, glue in a cobordism from (Λ_+, f_+) and $(\tilde{\Lambda}, \tilde{f})$, and then extend cylindrically. The same procedure can be performed on the negative end to obtain a stretched cobordism.

Let $\underline{l_+}$ be the smallest Reeb height of (Λ_+, f_+). Consider a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that for some fixed value A,

$$\rho(s) = \begin{cases} 1, & s \leq 0 \\ \frac{4\mu_{\mathcal{D}}}{t_+}, & s \geq A \end{cases}$$

Then $\gamma : M \times \mathbb{R}^N \times \mathbb{R} \to J^1M$ given by $\gamma(x, \eta, s) = \rho(s) f(x, \eta)$ is a homotopy of generating familes and for each s,

$$j_{\gamma_s}(x, \eta) = \left(x, \rho(s) \frac{\partial f}{\partial x}(x, \eta), \rho(s) f(x, \eta) \right)$$

is a Legendrian with generating family $f_s = \rho(s) f$. We will show later in Lemma 11 that this homotopy of generating families induces an embedded Lagrangian cobordism. However, this homotopy of generating families also induces an isotopy of Legendrians between (Λ_+, f_+) and $(\tilde{\Lambda}, \tilde{f} = f_A)$ with smallest Reeb height $\underline{\tilde{l}_+} > \frac{4\mu_{\mathcal{D}}}{t_+}$. Since Legendrian isotopy induces Lagrangian cobordism (see, for example, [13]), there exists an embedded GF-compatible Lagrangian concordance $(\mathcal{L}', F')$ between (Λ_+, f_+) and $(\tilde{\Lambda}, \tilde{f})$.

To form the desired top-stretched cobordism, remove the region of $\mathcal{L}$ with $t \in [t_+, \infty)$, replace it with $\mathcal{L}'$ and then extend cylindrically. Since $\mathcal{L}'|_{[t_-, t_+]} = \mathcal{L}|_{[t_-, t_+]}$, there are no additional critical points of Δ in this region. Thus,

$$4\mu_{\mathcal{D}}(\tilde{F}) = 4\mu_{\mathcal{D}}(F) < \underline{\tilde{l}_+}t_+,$$

and hence $(\tilde{\mathcal{L}}, \tilde{F})$ is a top-stretched cobordism from (Λ_-, f_-) to $(\widetilde{\Lambda_+}, \widetilde{f_+})$.

□

We now have a good understanding of the critical points and values of Δ_0 with $t \in [t_-, t_+]$. In order to capture information about the critical points that live outside this region, we will adjust the difference function Δ_0 to a *sheared* difference function Δ by adding a **Hamiltonian shearing function** H, which takes the form $H : \mathbb{R}_+ \to \mathbb{R}$ where

$$H(t) = \begin{cases} \frac{r_-}{2}(t - t_-)^2, & t \leq u_- \\ 0, & t \in [t_-, t_+] \\ -\frac{r_+}{2}(t - t_+)^2, & t \geq u_+. \end{cases}$$

See Figure 16. The constant r_+ will determine the slope of the derivative of the quadratic portion of H and the constant u_+ will determine the length of a “transition zone” where H will change from flat to quadratic.

Definition 8. Define the following constants, $r_\pm$, $u_\pm$ as follows:

(i) r_+ is chosen sufficiently large such that $r_+ > \overline{l_+}t_+ (> \underline{l_+}t_+ > 4\mu_{\mathcal{D}})$;

(ii) r_- is chosen sufficiently large such that $r_- < \underline{l_-}t_-$;

(iii) $u_\pm$ are chosen such that $\frac{2\mu_{\mathcal{D}}}{r_\pm} < \min\left\{\left|t_\pm^2 - u_\pm^2\right|, |t_\pm - u_\pm|^2\right\} < \frac{\underline{l_\pm}t_\pm}{2r_\pm}$.

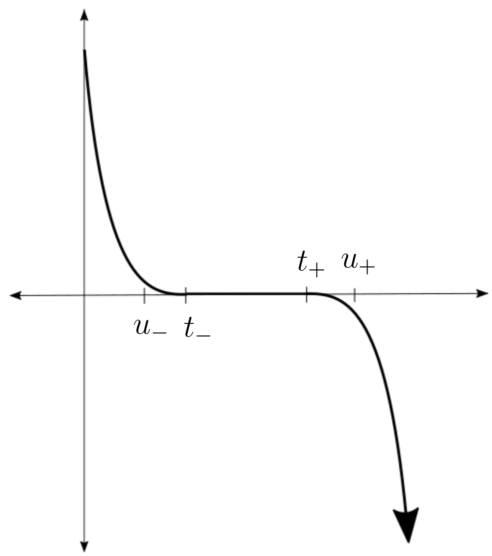


Figure 16: The Hamiltonian shearing function H(t). Note that H is smooth, decreasing, equal to 0 when $t \in [t_+, t_-]$, and quadratic outside of the region $[u_-, u_+]$.

Remark 7. 1. By the construction of a top-stretched filling, $\underline{l_+} t_+ > 4\mu_{\mathcal{D}}$ and hence $\frac{2\mu_{\mathcal{D}}}{r_+} < \frac{\underline{l_+} t_+}{2r_+}$.

2. The lower bound in Inequality (iii) is necessary in defining μ below and the upper bound is useful in the proof of Lemma 9(ii).

With r_+ and u_+ set, observe that the following inequalities hold:

$$t_\pm \underline{l_\pm} + \frac{\underline{l_\pm}^2}{2r_\pm} > t_\pm \underline{l_\pm} > \mu_{\mathcal{D}},$$

$$\frac{r_+}{2}(u_+ - t_+)^2 > \mu_{\mathcal{D}},$$

$$\frac{r_\pm}{2}\left|u_\pm^2 - t_\pm^2\right| > \mu_{\mathcal{D}},$$

$$\frac{u_+\underline{\underline{l_+}}}{2} > \frac{t_+\underline{\underline{l_+}}}{2} > \mu_{\mathcal{D}}.$$

Consequently, we are able to fix a large constant Ω and small constant μ so that all critical values of Δ with $t \in [t_-, t_+]$ lie within the region $[-\mu, \Omega]$.

Definition 9. Define the following constants, μ, and Ω as follows:

(i) μ is chosen to satisfy $\mu_{\mathcal{D}} < \mu < \min\left\{ \underbrace{t_-\underline{l_-}}_{1}, \underbrace{t_+\underline{l_\pm} + \frac{\underline{\underline{l_+}}^2}{2r_+}}_{2}, \underbrace{\frac{r_+}{2}(u_+ - t_+)^2}_{3}, \underbrace{\frac{u_+\underline{\underline{l_+}}}{2}}_{4}, \underbrace{\frac{r_\pm}{2}\left|u_\pm^2 - t_\pm^2\right|}_{5} \right\}$;

(ii) Ω is chosen sufficiently large such that $\Omega > \max\left\{ \underbrace{t_\pm\overline{l_\pm} + \frac{\overline{l_\pm}^2}{2r_\pm}}_{1}, \underbrace{\frac{\overline{l_+}}{u_+} - \frac{r_+}{2}(u_+ - t_+)^2}_{2} \right\}$.

Remark 8. In (i), Inequality 1 is needed in the proof of Lemma 10. Inequality 2 ensures that μ is less than all critical values coming from Reeb chords. Inequality 3 is needed in the proof of Lemma 9. Inequalities 4 and 5 are needed in the proof of Lemma 5.11 in [27] which allow us to identify terms in the long exact sequences.

In (ii), Inequality 1 is chosen so that Ω is larger than all critical values of Δ. This is needed in the proofs of Lemma 8.3 in [27]. Inequality 2 is needed in the proof of Lemma 9, to ensure that $\lambda_\Omega(u_+) > \overline{l_+}$.

With these values in hand, the **sheared difference function**, $\Delta_H : \mathbb{R}_+ \times M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\Delta_H(t, x, \eta, \tilde{\eta}) = F(t, x, \tilde{\eta}) + H(t) - F(t, x, \eta). \tag{16}$$

Remark 9. As shown in [27], if the pair (F, f) is tame, then the associated difference functions are tame. Furthermore, if (L, F) is a GF-compatible cobordism from

(Λ_-, f_-) to (Λ_+, f_+) then for any H, the triple $(\Delta_H, \delta_-, \delta_+)$ is equivalent to a tame triple of functions.

For ease of notation, we will simplify the notation of this sheared difference function to Δ if the choice of shearing function is clear from context. In the following theorem, we classify all critical points of Δ.

Theorem 10. *Suppose $(\Lambda_-, f_-) \prec_{(\mathcal{L},F)} (\Lambda_+, f_+)$. Then, there is a one-to-one correspondence between*

(i) The critical points of Δ in the region $t \in (-\infty, u_-) \cup (u_+, \infty)$ and the Reeb chords, $\gamma_\pm$ of $\Lambda_\pm$. We will refer to this set of critical points as $\mathcal{R}$. The critical value of the critical point corresponding to Reeb chords $\gamma_\pm$ is

$$tl(\gamma_\pm) + \frac{1}{2r_\pm}(l(\gamma_\pm))^2. \tag{17}$$

(ii) The critical points of Δ in the region $t \in [t_-, t_+]$ and the elements of the set $\mathcal{D} \sqcup \mathcal{C}$, where

$$\mathcal{D} = \left\{(t, x, \eta, \tilde{\eta}) : (t, x, \eta) \neq (t, x, \tilde{\eta}) \in \Sigma_F \text{ and } \partial_F(t, x, \eta) = \partial_F(t, x, \tilde{\eta})\right\},$$

and

$$\mathcal{C} = \left\{(t, x, \eta, \eta) : (t, x, \eta) \in \Sigma_F \text{ and } t \in [t_-, t_+]\right\}.$$

The non-degenerate critical submanifold, $\mathcal{C}$ has critical value 0, and is diffeomorphic to the fiber critical set, Σ_F. For each immersed double point in $\mathcal{L}$, *there is a pair of critical points in $\mathcal{D}$ whose critical values are negatives of one another.*

The critical points of types (i) and (ii) make up all of the critical points of Δ. The non-degenerate critical submanifold, $\mathcal{C}$ has index N and for any critical point $(t, x, \eta, \tilde{\eta}) \in \mathcal{D}$ with index k, there is a critical point $(t, x, \tilde{\eta}, \eta) \in \mathcal{D}$ with index $(1 + m + 2N) - k$, where $m = \dim M$ and F is defined on $\mathbb{R}_+ \times M \times \mathbb{R}^{2N}$.

Proof. The proof of part (i) can be found in [27]. The correspondence in part (ii) follows from Lemma 8. It remains to compute the indices of the critical points.

To calculate the index of the critical submanifold $\mathcal{C}$, let $(t_0, x_0, \eta_0, \eta_0) \in \mathcal{C}$ be arbitrary. Since $(t_0, x_0, \eta_0) \in \Sigma_F$, we have

$$\frac{\partial F}{\partial \eta}(t_0, x_0, \eta_0) = 0.$$

Thus, η_0 is a critical point of the function $F_{(t_0,x_0)} : \mathbb{R}^N \to \mathbb{R}$ given by

$$F_{(t_0,x_0)}(\eta) = F(t_0, x_0, \eta).$$

By the Morse Lemma, there exist local coordinates $(\eta_1, ..., \eta_N)$ such that

$$F_{(t_0,x_0)}(\eta) = F_{(t_0,x_0)}(\eta_0) - \eta_1^2 - ... - \eta_k^2 + \eta_{k+1}^2 + ... + \eta_N^2.$$

Thus, the following equalities hold locally near (t_0, x_0, η_0):

$$\begin{aligned}\Delta(t,x,\eta,\eta) &= F(t,x,\eta) - F(t,x,\eta)\\ &= F_{(t_0,x_0)}(\eta) - F_{(t,x)}(\eta)\\ &= F_{(t_0,x_0)}(\eta_0) - \eta_1^2 - ... - \eta_k^2 + \eta_{k+1}^2 + ... + \eta_N^2\\ &\quad - \left(F_{(t_0,x_0)}(\eta_0) - \eta_1^2 - ... - \eta_k^2 + \eta_{k+1}^2 + ... + \eta_N^2\right)\\ &= -\eta_1^2 - ... - \eta_k^2 - \eta_{k+1}^2 - ... - \eta_N^2 + \eta_1^2 + ... + \eta_k^2 + \eta_{k+1}^2 + ...\eta_N^2.\end{aligned}$$

Therefore, $(t_0, x_0, \eta_0, \eta_0)$ has index N and hence the critical submanifold $\mathcal{C}$ has index N.

Now, let $(t_0, x_0, \eta_0, \tilde{\eta}_0)$ be a critical point of index i living in $\mathcal{D}$. Then, again by the Morse Lemma, there exist local coordinates $(y_1, ...y_{1+m+2N})$ such that

$$\Delta(t,x,\eta,\tilde{\eta}) = \Delta(t_0,x_0,\eta_0,\tilde{\eta}_0) - y_1 - ... - y_i + y_{i+1} + ... + y_{1+m+2N}.$$

Then, we have:

$$\begin{aligned}\Delta(t,x,\tilde{\eta},\eta) &= -\Delta(t,x,\eta,\tilde{\eta})\\ &= -\Delta(t_0,x_0,\eta_0,\tilde{\eta}_0) + y_1 + ... + y_i - y_{i+1} - ... - y_{1+m+2N}\\ &= \Delta(t_0,x_0,\widetilde{\eta_0},\eta_0) + y_1 + ... + y_i - y_{i+1} - ... - y_{1+m+2N}.\end{aligned}$$

Thus, $(t_0, x_0, \widetilde{\eta_0}, \eta_0)$ has index $1 + m + 2N - i$. This completes the proof. □

By restricting to stretched cobordisms, the positive constants Ω and μ were chosen in such a way that all critical points of Δ with $t \in [t_-, t_+]$ lie in $[-\mu, \mu]$ and all critical values of Δ arising from Reeb chords lie in $[\mu, \Omega]$. Keeping in mind that we will use Morse theory on Δ, we will define the wrapped generating family homology groups on F in terms of sublevel sets.

Definition 10. For any $a \in \mathbb{R}$, the **sublevel set of** Δ is given by

$$\Delta^a = \{(t, x, \eta, \tilde{\eta}) : \Delta(t, x, \eta, \tilde{\eta}) \leq a\}.$$

and the **sublevel set of** Δ **restricted to a region** $[i, j] \subset \mathbb{R}$ is given by

$$\Delta^a_{[i,j]} = \{(t, x, \eta, \tilde{\eta}) : t \in [i, j], \Delta(t, x, \eta, \tilde{\eta}) \leq a\}.$$

Due to the fact that $\mathcal{L}$ is cylindrical after a certain value for t, critical values of Δ within this region can be identified with critical values of δ. It is therefore useful to define the function below, which translates a critical value of Δ into the corresponding critical value of δ.

Definition 11. Define the (Δ, δ)**-translator function,** $\lambda_a(t) : \mathbb{R} \to \mathbb{R}$ by

$$\lambda_a(t) = \frac{1}{t}(a - H(t)).$$

Observe that for $[i, j] \subset [t_+, \infty)$,

$$\Delta^a_{[i,j]} = \{(t, x, \eta, \tilde{\eta}) : t \in [i, j], \delta_+(x, \eta, \tilde{\eta}) \leq \lambda_a(t)\}.$$

To summarize the analysis in this section, we have shown that the sublevel sets of Δ within $[u_-, u_+]$ capture the critical sumbanifold and immersed double points of $\mathcal{L}$ and the sublevel sets outside of this region capture the Reeb chords of the cylindrical Legendrian ends. At this point, one should be convinced that it makes sense to consider the following homology groups for stretched cobordisms:

Definition 12. Suppose $(\mathcal{L}, F)$ is a stretched GF-compatible cobordism from (Λ_-, f_-) to (Λ_+, f_+). The **total wrapped generating family cohomology** of F is given by

$$\widetilde{WGH}^k(F) = H^{k+N}(\Delta^\Omega, \Delta^{-\mu}).$$

The **relative wrapped generating family cohomology** of F is given by

$$WGH^k(F) = H^{k+N}(\Delta^\Omega, \Delta^\mu).$$

Remark 10. In [27], the same name is given to the analogous homology groups of embedded GF-compatible Lagrangian cobordisms. Since these homology groups of immersed stretched cobordisms are defined in a similar way, we use the same name.

Remark 11. For tame (F, f_-, f_+), $\widetilde{WGH}^k$ and WGH^k do not depend on choice of Ω and μ, as proven in [27]. The proof relies on Lemma 3 below and uses a Morse-theoretic argument similar to the one used to show independence of ω and ϵ.

The following Lemmas of [27] will be useful in proving Theorem 1.

Lemma 3. *(Corollary 4.10 in [27]) There exist values $v_- < t_-$ and $v_+ > t_+$ such* *that*

$$WGH^k(F) = H^{k+N}\left(\Delta^{\Omega}_{[v_-,v_+]}, \Delta^{\mu}_{[v_-,v_+]}\right)$$

and

$$\widetilde{WGH}^k(F) = H^{k+N}\left(\Delta^{\Omega}_{[v_-,v_+]}, \Delta^{-\mu}_{[v_-,v_+]}\right)$$

Lemma 4. *(Proposition 4.12 in [27])* $\widetilde{WGH}^k(F) = 0$.

Chapter 5

Mapping Cones and Gradient Flows

5.1 Mapping Cone Background

The proof of Theorem 1 will follow from the following theorem which equates the generating family cohomology groups of the Legendrian to Morse cohomology groups associated to the Lagrangian filling:

Theorem 11. *Suppose* (Λ_+, f_+) *admits an immersed GF-compatible filling* $(\mathcal{L}, F)$*. Then*

$$GH^k(\Lambda_+, f_+) \cong H^{k+N+1}\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right). \tag{18}$$

The proof of this theorem follows a similar structure to that in [27]. We show that the total space, $\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right)$, whose cohomology vanishes, can be viewed as a mapping cone. Then we show that the cohomology groups in (18) fit into a long exact sequence involving this mapping cone. To that end, we now recall the following definition and lemma.

Definition 13. Let (X, A), (Y, B) be pairs, and let $\phi : (X, A) \to (Y, B)$ be a map between the pairs. Let I denote the unit interval $[0, 1]$. The **relative mapping cone** $C(X, A)$ **of** (X, A) is the pair $(X \times I, A \times I \cup X \times \{1\})$. The **relative mapping cone** $C(\phi)$ **of** ϕ is the pair $C(X, A) \cup_\phi (Y, B)$, where $\cup_\phi$ denotes the identification of $(x, 0)$ with $\phi(x)$.

Lemma 5 (Lemma 5.3 in [27]). *Let $\phi : (X, A) \to (Y, B)$ be a map between pairs. Then the following long exact sequence exists:*

$$... \to H^k(C(\phi)) \to H^k(Y, B) \to H^k(X, A) \to ...$$

The proof of this lemma can be found in [27]. For the reader's convenience, it is included below.

Proof. From the triple,

$$(c, b, a) := ((X \times I) \cup_\phi Y, (A \times I \cup X \times \{1\}) \cup_\phi Y, (A \times I \cup X \times \{1\}) \cup_\phi B),$$

we obtain the following long exact sequence:

$$... \to H^k(c, b) \to H^k(c, a) \to H^k(b, a) \to ...$$

Excising $(A \times I \cup X \times \{1\})$ from (b, a), we get

$$H^k(b, a) \cong H^k(Y, B).$$

In addition, we have

$$\begin{aligned} H^k(c, a) &\cong H^k(X \times I, A \times I \cup X \times \{1\}) \cup_\phi (Y, B)) \\ &\cong H^k(C(X, A) \cup_\phi (Y, B)) \\ &\cong H^k(C(\phi)). \end{aligned}$$

Finally, by collapsing Y to a point, we get

$$\begin{aligned} H^k(c, b) &\cong H^k(X \times I, A \times I \cup X \times \{0, 1\}) \\ &\cong H^k(\Sigma(X, A)) \\ &\cong H^{k-1}(X, A), \end{aligned}$$

giving us the desired sequence. □

The following lemmas from [27] will be useful in identifying pairs of sublevel sets with relative cones.

Lemma 6 (Lemma 5.6 in [27]). *Let $\delta : X \to \mathbb{R}$ be a smooth function whose negative gradient flow exists for all time. Let $a, b : J = [t_0, t_1] \to \mathbb{R}$ be continuous functions satisfying the following:*

1. *$b(t) = b(t_0)$ for all t,*

2. $a(t)$ *strictly increasing with* $a(t_1) = b(t_0)$*,*

3. $a(t_0)$ *has a neigborhood of regular values of* δ*.*

Then $(B_J, A_J) = \left(\bigcup_{t\in J}\{t\} \times \delta^{b(t)}, \bigcup_{t\in J}\{t\} \times \delta^{a(t)}\right)$ *deformation retracts onto* $C\left(\delta^{b(t_0)}, \delta^{a(t_0)}\right)$*.*

Remark 12. To prove this theorem, Sabloff-Traynor define a map σ that is homotopic to the identity map on B_J and follows its negative gradient flow. For the reader's convenience, we include the details of the proof below. Schematic pictures of σ and its image are pictured in Figures 17 and 18.

Proof. Fix $0 < \epsilon < b(t_0) - a(t_0)$ such that no critical values lie in the region $[a(t_0) - \epsilon, a(t_0) + \epsilon]$. Consider the strictly increasing straight-line function $\alpha(t)$ such that $\alpha(t_0) = a(t_0)$ and $\alpha(t_1) = a(t_0) + \epsilon$. Define a map $\sigma : B_J \to B_J$ as follows:

$$\sigma(t,x) = \begin{cases} (t,x), & \delta(x) \leq \alpha(t) \\ (\alpha^{-1}(\delta(x)), x), & \alpha(t) \leq \delta(x) \leq \alpha(t_1) \\ (t_1, x), & \alpha(t_1) \leq \delta(x) \leq a(t) \\ *, & a(t) \leq \delta(x) \leq a(t) + \epsilon \\ (t,x), & \delta(x) \geq a(t) + \epsilon. \end{cases}$$

*On this region, σ interpolates between the two extremes. (See Figure 17.) Following the flow of the horizontal vector field ∂_t gives a homotopy of σ to the identity map on B_J. Following the negative gradient flow of δ gives a map from $(\sigma(B_J), \sigma(A_J))$ to $\left(J \times \delta^{b(t_0)}, J \times \delta^{a(t_0)} \cup \{t_1\} \times \delta^{b(t_1)}\right) = C\left(\delta^{b(t_0)}, \delta^{a(t_0)}\right)$. □

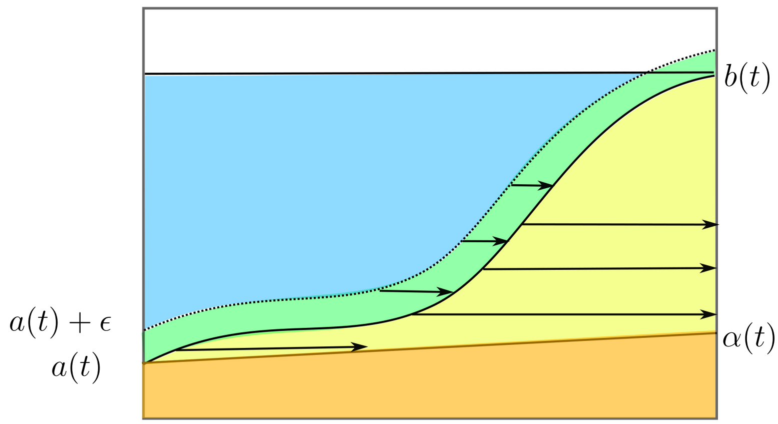


Figure 17: Schematic picture of σ.

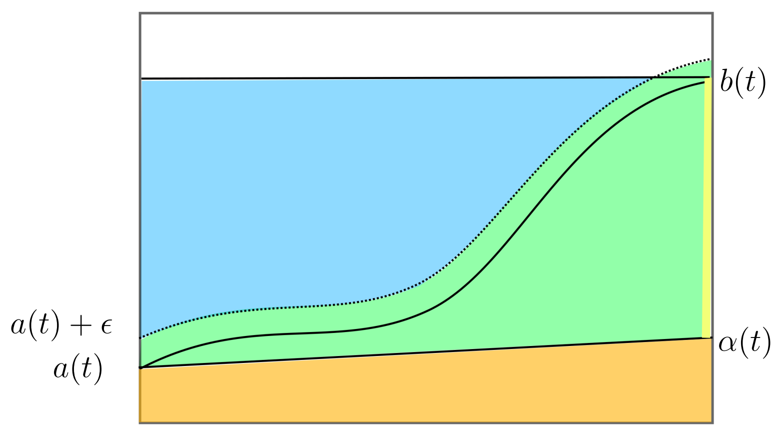


Figure 18: Schematic picture of the image of σ.

Lemma 7. *(Corollary 5.5 in [27]) Let $\delta : X \to \mathbb{R}$ be a continuous function and let $a, b : J = [t_0, t_1] \to \mathbb{R}$ be continuous functions satisfying the following:*

1. *$a(t) \leq b(t)$ for all t, and*

2. *$a(t)$ and $b(t)$ are strictly increasing.*

Then $(B_J, A_J) = \left(\bigcup_{t\in J}\{t\} \times \delta^{b(t)}, \bigcup_{t\in J}\{t\} \times \delta^{a(t)}\right)$ deformation retracts onto (B_{t_1}, A_{t_1}).

Remark 13. To define an appropriate deformation retraction, one first follows the horizontal gradient flow, taking (B_J, A_J) to $(A_J \cup B_{t_1}, A_J)$ and then retracting $(A_J \cup B_{t_1}, A_J)$ onto (B_{t_1}, A_{t_1}) under the map $(t, x) \mapsto (t_1, x)$.

5.2 Analysis of Sublevel Sets

In the proofs to follow, it will often be convenient to understand the negative gradient flow of Δ at certain values of t. These are summarized in the lemma below and pictured in Figure 19.

Lemma 8. *Set constants*

$$\sigma_\pm = r_\pm u_\pm |u_\pm - t_\pm| \pm \frac{r_\pm}{2} (u_\pm - t_\pm)^2 .$$

The gradient flow of Δ satisfies the following:

1. *$\partial_t \Delta < 0$ on $\{v_-\} \cap \{\mu < \Delta < \Omega\}$;*

2. *$\partial_t \Delta < 0$ on $\{u_-\} \cap \{\mu < \Delta < \sigma_-\}$;*

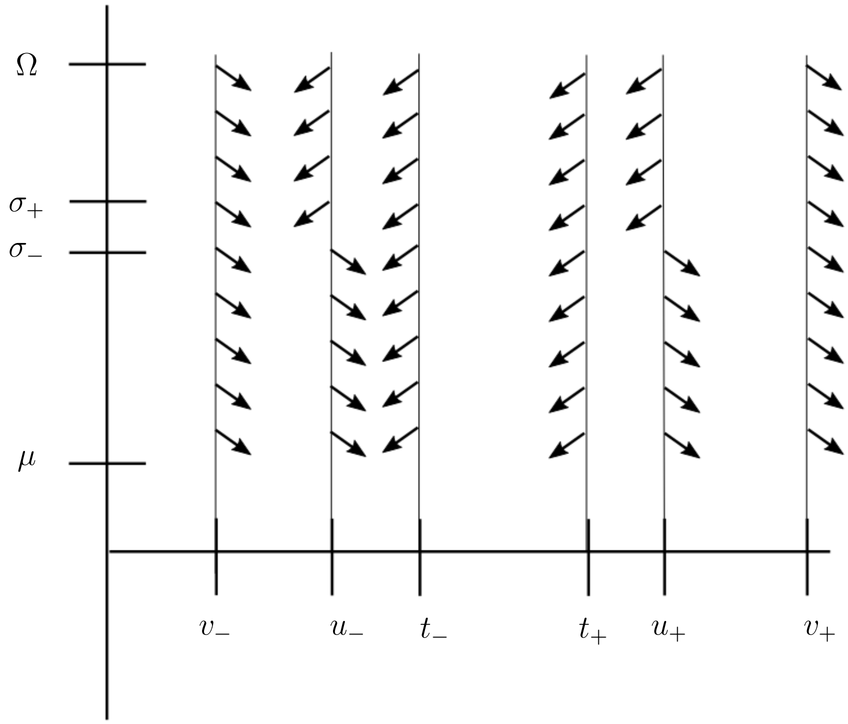


Figure 19: Negative gradient flow of Δ.

3. $\partial_t \Delta > 0$ *on* $\{u_-\} \cap \{\sigma_- < \Delta < \Omega\}$*;*

4. $\partial_t \Delta > 0$ *on* $\{t_-\} \cap \{\mu < \Delta < \Omega\}$*;*

5. $\partial_t \Delta < 0$ *on* $\{v_+\} \cap \{\mu < \Delta < \Omega\}$*;*

6. $\partial_t \Delta < 0$ *on* $\{u_+\} \cap \{\mu < \Delta < \sigma_+\}$*;*

7. $\partial_t \Delta > 0$ *on* $\{u_+\} \cap \{\sigma_+ < \Delta < \Omega\}$*;*

8. $\partial_t \Delta > 0$ *on* $\{t_+\} \cap \{\mu < \Delta < \Omega\}$*.*

Proof. Recall that when $t \geq t_+$ or $t \leq t_-$,

$$\partial_t \Delta = \delta_\pm + H'(t),$$

and when $-\mu < \Delta < \Omega$, we also have for each t,

$$\delta_\pm < \lambda_\Omega(t) \text{ and } \delta_\pm > \lambda_{-\mu}(t).$$

1. Since v_- was chosen so that $\lambda_\Omega(v_-) < -\overline{l_-} < 0$ and $H'(v_-) < 0$, we have $\partial_t \Delta|_{t=v_-} = \delta_- + H'(v_-) < 0$.

2. At $t = u_-$, $\partial_t \Delta|_{\{t=u_-\}} = \delta_- + r_-(u_- - t_-)$. Since

$$\delta_- < \lambda_\sigma(u_-) = \frac{1}{u_-}\left(\sigma - \frac{r_-}{2}(u_- - t_-)^2\right) = r_-|u_- - t_-|,$$

   it follows that $\partial_t \Delta|_{\{t=u_-\}} < 0$.

3. Similarly, since

$$\delta_- > \lambda_\sigma(u_-) = \frac{1}{u_-}\left(\sigma - \frac{r_-}{2}(u_- - t_-)^2\right) = r_-|u_- - t_-|,$$

   it follows that $\partial_t \Delta|_{\{t=u_-\}} > 0$.

4. At t_-, we have $\partial_t \Delta|_{\{t=t_-\}} = \delta_- \geq \lambda_\mu(t_-) = \frac{\mu}{t_-} > 0$.

5. - 8. (Similar.)

□

With Lemma 5 in hand, we now seek a map between pairs that relate to the cohomology groups in Theorem 11. A natural map to consider is the inclusion map

$$\phi : \left(\Delta^{\Omega}_{\{u_+\}}, \Delta^{-\mu}_{\{u_+\}}\right) \to \left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right).$$

We first show that the cohomology groups $H^k\left(\Delta^{\Omega}_{\{u_+\}}, \Delta^{-\mu}_{\{u_+\}}\right)$ coincide with the generating family cohomology of the Legendrian. From this, we will also be able to identify the mapping cone of ϕ with the pair $\left(\Delta^{\Omega}_{[v_-,v_+]}, \Delta^{-\mu}_{[v_-,v_+]}\right)$.

Lemma 9 (Lemma 6.2 in [27])**.** *There exists a diffeomorphism*

$$\left(\Delta^{\Omega}_{\{u_+\}}, \Delta^{-\mu}_{\{u_+\}}\right) \cong \left(\delta^{\omega}_{+}, \delta^{\epsilon}_{+}\right),$$

and a retract

$$\rho : \left(\Delta^{\Omega}_{[u_+,v_+]}, \Delta^{-\mu}_{[u_+,v_+]}\right) \to C\left(\delta^{\omega}_{+}, \delta^{\epsilon}_{+}\right).$$

Remark 14. The first statement follows exactly from the definitions of $\Delta^{\Omega}_{\{u_+\}}$ and $\lambda_{\Omega}(u_+)$. The proof of the second statement is an application of Lemma 6 with $a, b : [u_+, v_+] \to \mathbb{R}$ given by $a(t) = \lambda_{-\mu}(t)$ and $b(t) = \lambda_{\Omega}(t)$. Some work is required to show that a and b satisfy the necessary conditions of Lemma 6. We include the details below for the reader's convenience.

Proof. To prove the first statement, observe that

$$\begin{aligned}\Delta^{\Omega}_{\{u_+\}} &= \{(u_+, x, \eta, \tilde{\eta}) : \Delta(u_+, x, \eta, \tilde{\eta}) < \Omega\} \\ &\cong \{(x, \eta, \tilde{\eta}) : \delta_+(x, \eta, \tilde{\eta}) < \lambda_\Omega(u_+)\} \\ &= \delta_+^{\omega},\end{aligned}$$

since $\lambda_\Omega(u_+) > \overline{l_+}$ by (ii.2) in Definition 9.

The proof of the second statement will be an application of Lemma 6. Consider the paths $a, b : [u_+, v_+] \to \mathbb{R}$ given by $a(t) = \lambda_{-\mu}(t)$ and $b(t) = \lambda_\Omega(t)$, as defined in Definition 10. To apply Lemma 6, we must show that $a(t)$ is increasing and has a neighborhood of regular values, and that $b(t)$ can be identified with a constant function equal to $a(v_+)$.

To see that $a(t)$ is increasing, first note that for $t \in (t_+, v_+]$, $H(t), H'(t)$, and $H''(t)$ are all negative. Then, for t in this region, we have

$$t^2\lambda'_{-\mu}(t) = \mu - tH'(t) + H(t), \text{ and}$$

$$t^2\lambda'_{-\mu}(t_+) = \mu > 0.$$

Since

$$\left(t^2\lambda'_{-\mu}\right)'(t) = -tH''(t) > 0,$$

$\lambda'_{-\mu}$ is increasing and thus positive for all $t \in [t_+, v_+]$. Therefore, $a(t) = \lambda_{-\mu}(t)$ is increasing on this region as well.

To show that $a(u_+)$ has a neighborhood of regular values we will show that it is positive and strictly less than $\underline{l_+}$. We have

$$\begin{aligned}
a(u_+) &= \frac{1}{u_+}\left(-\mu - H(u_+)\right) \\
&= \frac{1}{u_+}\left(-\mu + \frac{r_+}{2}(u_+ - t_+)^2\right), \text{ which is positive by Definition 9(i.3),} \\
&\leq \frac{1}{u_+}\left(-\mu + \frac{1}{2}\left(\frac{\underline{l_+}t_+}{2|u_+ - t_+|^2}\right)(u_+ - t_+)^2\right) \text{ by Definition 8(iii),} \\
&= -\frac{\mu}{u_+} + \frac{\underline{l_+}t_+}{4u_+} \\
&< \underline{l_+},
\end{aligned}$$

as desired.

In order to identify $b(t)$ with $a(v_+)$, first note that by definition of v_+, $a(v_+)$ is greater than all critical values of δ_+. We will show that $b(t)$ is also greater than all critical values of δ. Then, since there are no critical values between $b(t)$ and $a(v_+)$ for $t \in [t_+, v_+]$, we can follow the gradient flow of δ to redefine $b(t) = a(v_+)$ for all $t \in [t_+, v_+]$.

To show that $b(t)$ is greater than all critical values of δ, we will show that there is a unique minimum t_+^m on $[t_+, \infty)$ such that $\lambda_\Omega(t_+^m) > \overline{l_+}$. Since $H'(t)$ is unbounded below and decreasing on $[t_+, \infty)$, there is a unique point $\overline{t_+}$ such that $H'\left(\overline{t_+}\right) = -\overline{l_+}$. Now,

$$\begin{aligned}
\overline{t_+}^2 \lambda_\Omega'\left(\overline{t_+}\right) &= -\overline{t_+} H'\left(\overline{t_+}\right) - \Omega + \frac{\overline{l_+}^2}{2r_+} \\
&= \overline{t_+}\,\overline{l_+} - \Omega + \frac{\overline{l_+}^2}{2r_+} < 0,
\end{aligned}$$

by Inequality (ii.1) in Definition 9 of Ω. On the other hand, for sufficiently large t,

$$t^2\lambda'_\Omega(t) > 0,$$

and thus there exists t_+^m such that $t^2\lambda'_\Omega\left(t_+^m\right) = \lambda'_\Omega\left(t_+^m\right) = 0$. Since λ'_Ω is increasing, t_+^m is a unique minimum on this region. Now, since

$$\left(t_+^m\right)^2 \lambda'_\Omega\left(t_+^m\right) = \Omega - t_+^{\bar{m}} H'\left(t_+^m\right) + H\left(t_+^m\right) = 0,$$

we have

$$\begin{aligned}\lambda_\Omega\left(t_+^m\right) &= \frac{1}{t_+^m}\left(\Omega - H(t_+^m)\right)\\ &= \frac{1}{t_+^m}\left(\Omega - \left(-t_+^m H'((t_+^m) - \Omega\right)\right)\\ &= -H'\left(t_+^m\right) > H'\left(\overline{t_+}\right) = \overline{l_+},\end{aligned}$$

as desired.

With all conditions of Lemma 6 satisfied, we can conclude that $\left(\Delta^\Omega_{[u_+,v_+]}, \Delta^{-\mu}_{[u_+,v_+]}\right)$ retracts to $C\left(\delta_+^{b(u_+)}, \delta_+^{a(u_+)}\right) = C\left(\delta_+^\omega, \delta_+^\epsilon\right)$. □

Corollary 2. *The pair* $\left(\Delta^\Omega_{[v_-,v_+]}, \Delta^{-\mu}_{[v_-,v_+]}\right)$ *is the mapping cone of the map*

$$\phi : \left(\Delta^\Omega_{\{u_+\}}, \Delta^{-\mu}_{\{u_+\}}\right) \to \left(\Delta^\Omega_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right),$$

where ϕ *is given by inclusion.*

Proof. By definition,

$$C(\phi) = C\left(\Delta^{\Omega}_{\{u_+\}}, \Delta^{-\mu}_{\{u_+\}}\right) \cup_{\phi} \left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right),$$

which, by Lemma 9, is homotopy equivalent to

$$\left(\Delta^{\Omega}_{[u_+,v_+]}, \Delta^{-\mu}_{[u_+,v_+]}\right) \cup_{\phi} \left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right) = \left(\Delta^{\Omega}_{[v_-,v_+]}, \Delta^{-\mu}_{[v_-,v_+]}\right).$$

□

Chapter 6

Proofs of Obstruction Theorems

The proofs of Theorem 11, Theorem 1, and Corollary 1 are now straightforward applications of the above Lemmas, which are datailed below.

6.1 Filling Obstructions

Proof of Theorem 11. Let Λ be a Legendrian in J^1M with linear-at-infinity generating family f_+. Suppose (Λ_+, f_+) has an immersed, GF-compatible, top-stretched Lagrangian filling $(\mathcal{L}, F)$ in $\mathbb{R} \times J^1M$.

Letting $v_\pm$ and ϕ be defined as above, we have the following long exact sequence:

$$\cdots \to H^k(C(\phi)) \to H^k\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right) \to H^k\left(\Delta^{\Omega}_{\{u_+\}}, \Delta^{-\mu}_{\{u_+\}}\right) \to \cdots .$$

By Lemma 9 and the definition of generating family cohomology of Λ_+,

$$H^k\left(\Delta^{\Omega}_{\{u_+\}}, \Delta^{-\mu}_{\{u_+\}}\right) \cong H^k\left(\delta^{\omega}_+, \delta^{\epsilon}_+\right) \cong GH^{k-N-1}(\Lambda_+, f_+).$$

By Lemmas 2 and 4

$$H^k(C(\phi)) \cong H^k\left(\Delta^{\Omega}_{[v_-,v_+]}, \Delta^{-\mu}_{[v_-,v_+]}\right) = \widetilde{WGH}^{k-N-1}(F) = 0.$$

Thus, we get the isomorphism

$$H^{k+N+1}\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right) \cong GH^k(\Lambda_+, f_+),$$

as desired.

□

Proof of Theorem 1. We shall show that

$$H^{k+N+1}\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right) \cong H_{n-k}\left(C(\Sigma_F, \{x_i\}), d_*\right), \tag{19}$$

where n is the dimension of Σ_F and d_* is defined by the gradient flow between the corresponding critical points of Δ. First note that by Poincaré duality,

$$H^{k+N+1}\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right) \cong H_{N+m-k}\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right).$$

Thus, for some sequence of nonnegative integers x_i, Theorem 11 implies that Δ will have

- a critical submanifold of index N,
- x_i critical points of index $N+m-i$ and of index $N+m+i$.

Similarly, since Σ_F is dimension $m+1$, $C\left(\Sigma_F, \{x_i\}\right)$ has

- $\dim H^k\left(\Sigma_F\right)$ generators of index $k+1$ for each $k \in \{0, ..., m\}$,
- x_i generators of index i and of index $-i$.

My standard Morse-Bott theory, perturbing the index N critical submanifold of Δ will give rise to a set of critical points whose indices lie within the range $[N, N+m+1]$. These correspond exactly to the generators of $C(\Sigma_F, \{x_i\})$ in the first bullet point shifted by N.

To show that Equation 19 holds, we will first show that the homology groups of $\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right)$ agree with those of $\left(\Delta^{\sigma_+}_{[t_-,t_+]}, \Delta^{-\mu}_{[t_-,t_+]} \cup \Delta^{\sigma_+}_{\{t_+\}}\right)$ using an argument similar to that in the Proof of Lemma 6.5 in [27]. Since the critical submanifold $\mathcal{C}$ is properly embedded in $\mathbb{R}_+ \times M \times \mathbb{R}^N \times \mathbb{R}^N$ and since Δ has no critical points when $t = t_\pm$, there is a choice of metric that allows us to assume the gradient flow of Δ is tangent along this boundary. The Morse-Bott Lemma will then allow us to identify the homology groups in Equation 19.

In order to show that the homology groups of $\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right)$ and $\left(\Delta^{\sigma_+}_{[t_-,t_+]}, \Delta^{-\mu}_{[t_-,t_+]} \cup \Delta^{\sigma_+}_{\{t_+\}}\right)$ agree, we will first show that $\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right)$ deformation retracts onto $\left(\Delta^{\sigma_+}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right)$. This is achieved by flowing along the negative gradient vector field of Δ on $[v_-, u_+] \times M \times \mathbb{R}^{2N}$ and stopping when the value of Δ reaches σ_+. Since $\sigma_+ > \mu$, the top-stretched condition implies that there are no critical values Δ within $[\sigma_+, \Omega]$. Numbers 1 and 7 in Lemma 8 show that the negative gradient flow is inward pointing at the boundaries.

Next, consider $\left(\Delta^{\sigma+}_{[v_-,t_-]}, \Delta^{-\mu}_{[v_-,t_-]}\right)$. Applying Lemma 7 with $a(t) = \lambda_\Omega(t)$ and $b(t) = \lambda_{-\mu}(t)$ shows that this deformation retracts onto $\left(\Delta^{\sigma+}_{\{t_-\}}, \Delta^{-\mu}_{\{t_-\}}\right)$.

Finally, consider $\left(\Delta^{\sigma+}_{[t_+,u_+]}, \Delta^{-\mu}_{[t_+,u_+]}\right)$. Applying Lemma 6 with $a(t) = \lambda_{-\mu}(t)$ and $b(t) = \lambda_\Omega(t)$ on the interval $[t_+, u_+]$, produces a deformation retract onto the cone space:

$$\left(\Delta^{\sigma+}_{\{t_+\}} \times [t_+, u_+], \left(\Delta^{-\mu}_{\{t_+\}} \times [t_+, u_+]\right) \cup \left(\Delta^{\sigma+}_{\{t_+\}} \times \{u_+\}\right)\right).$$

Thus, we have a deformation retract of $\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right)$ onto:

$$\left(\Delta^{\sigma+}_{[t_-,t_+]} \times [t_+, u_+], \left(\Delta^{-\mu}_{[t_-,t_+]} \times [t_+, u_+]\right) \cup \left(\Delta^{\sigma+}_{\{t_+\}} \times \{u_+\}\right)\right).$$

Excising $[t_+, u_+]$, the cohomology groups of this space agree with those of:

$$\left(\Delta^{\sigma+}_{[t_-,t_+]}, \Delta^{-\mu}_{[t_-,t_+]} \cup \Delta^{\sigma+}_{\{t_+\}}\right).$$

□

Proof of Corollary 1. In the case that Λ is a Legendrian knot, $m = 1$. If Σ_F has genus g then the generators $C(\Sigma_F, \{x_i\})$ corresponding to the index N critical submanifold of Δ include one generator of index 1 and $2g$ generators of index 2. Suppose C_k denotes the kth chain in $C(\Sigma_F, \{x_i\})$. Then, by Theorem 1,

$$\dim GH^k(f_+) = \dim H_{2-k}\left(C(\Sigma_F, \{x_i\}), \partial\right) \leq \dim C_{2-k}.$$

Thus, $\dim GH^k(f_+)$ determines a lower bound for the minimal number of generators of C_{2-k}. Since $m = 1$, this is equal to the number of index $N+1 \pm k$ critical points of Δ. By Theorem 10, this corresponds to a minimal number of immersed double points of index $\pm k$. There is a special case when considering the index $k = 0$ of $GH^k(f_+)$. In this case, a pair of generators could either correspond to an immersion point of index 0 or additional genus.

□

In Chapter 7, we discuss methods of constructing immersed GF-compatible fillings, including ways of creating new fillings from existing ones. Before switching our focus to *constructions*, we provide a final *obstruction* to the existence of immersed GF-compatible fillings for Legendrian knots which justifies the lattice configuration of the diagrams in the introduction. The proof of this obstruction relies on the following classical result of J.H.C. Whitehead. Barannikov also gives a proof of this in [3].

Proposition 12. *Suppose C_* is an ordered chain complex, that is, for each k, the generators of C_k have a fixed ordering. Any ordered chain complex C_* is equivalent to an ordered chain complex $\widetilde{C_*}$ such that for each generator a in $\widetilde{C_k}$, either $da = 0$ or $da = b$ for a unique generator $b \in \widetilde{C_{k-1}}$.*

Remark 15. The notion of equivalence between ordered chain complexes C_* and $\widetilde{C_*}$ we refer to in the proposition is the following: for each k, C_k and $\widetilde{C_k}$ have the same dimension and their differentials coincide on coinciding generators. We assume that coefficients are taken over a field, F. Such an ordered basis is said to be of **canonical form over F**. The proof of this theorem can be found in Lemma 2 of [3]. For the reader's convenience, it is included below.

Proof. Let e_j^k denote a generator in C_* of index k such that one of the following is satisfied:

1. for $i = k$ and for all $m \leq j$, de_m^i has the required form; or

2. for all $i \leq k$ and for all m, de_m^i has the required form.

We shall show that both de_{j+1}^k and de_j^{k+1} can be adjusted to be of the proper form.

First, consider e_{j+1}^k. We will produce a new generator $\tilde{e}_{j+1}^k$ expressed in terms of $\{e_q^k\}_{q=1}^j$ that is of the required form. For some $\{\alpha_n\} \subset F$,

$$de_{j+1}^k = \sum_n \alpha_n e_n^{k-1}.$$

Rearrange this equation by moving any terms such that $e_n^{k-1} = de_q^k$, with $q \leq j$, to the left hand side. We then get the following:

$$d\left(e_{j+1}^k - \sum_{q=1}^j \alpha_{n(q)} e_q^k\right) = \sum_n \beta_n e_n^{k-1},$$

where where $\beta_n = 0$ if $e_n^{k-1} = de_q^k$ and $\beta_n = \alpha_n$ otherwise. For such n where $\beta_n = 0$, define the following:

$$\widetilde{e}_{j+1}^k = e_{j+1}^k - \sum_{q=1}^j \alpha_{n(q)} e_q^k.$$

To define $\widetilde{e}_{j+1}^k$ for all other values of n, choose a value $n_0 \neq n(q)$ for all $q \leq j$ such that $\beta_{n_0} \neq 0$, and

$$d\left(e_{j+1}^k - \sum_{q=1}^j \alpha_{n(q)} e_q^k\right) = \beta_{n_0} e_{n_0}^{k-1} + \sum_{n \leq n_0} \beta_n e_n^{k-1}.$$

Since $d^2 = 0$, $de_n^{k-1} = 0$ for all n with $\beta_n \neq 0$. Now define:

$$\widetilde{e}_{j+1}^k = \left(e_{j+1}^k - \sum_{q=1}^{j} \alpha_{n(q)} e_q^k \right) / \beta_{n_0},$$

and

$$\widetilde{e}_{n_0}^{k-1} = e_{n_0}^{k-1} + \sum_{n \leq n_0} (\beta_n / \beta_{n_0}) e_n^{k-1}.$$

Then

$$\begin{aligned} d(\widetilde{e}_{j+1}^k) &= d\left(\left(e_{j+1}^k - \sum_{q=1}^{j} \alpha_{n(q)} e_q^k \right) / \beta_{n_0} \right) \\ &= \left(\beta_{n_0} e_{n_0}^{k-1} + \sum_{n \leq n_0} \beta_n e_n^{k-1} \right) / \beta_{n_0} \\ &= \widetilde{e}_{n_0}^{k-1}, \end{aligned}$$

making $\widetilde{e}_{j+1}^k$ of the required form. A similar process can be preformed to construct $\tilde{e}_j^{k+1}$ so that $d\tilde{e}_j^{k+1}$ is of the proper form.

It remains to show uniqueness of $\widetilde{C_*}$. Suppose there exist two canonical forms of C_* over F. Let $\{a_j^k\}$ and $\{b_j^k\}$ be sets of ordered generators of C_k for these canonical forms. Assume that in one of the following cases:

1. for $i = k$ and for all $m \leq j$; or

2. for all $i \leq k$ and for all m,

$da_m^i = a_n^{i-1}$ implies $db_m^i = b_n^{i-1}$, in other words, the canonical forms coincide. Suppose that $da_j^k = a_t^{k-1}$ and $db_j^k = b_l^{k-1}$. Without loss of generality, assume $t > l$. Since $\{a_j^k\}$

is an ordered basis for C_k, there exists $\{\alpha_n\}, \{\beta_n\} \subset F$ such that

$$b_j^k = \sum_{n=1}^{j} \alpha_n a_n^k,$$

and

$$b_l^{k-1} = \sum_{n=1}^{l} \beta_n a_n^{k-1}.$$

Then $d\left(\sum_{n=1}^{j} \alpha_n a_n^k\right) = \sum_{n=1}^{l} \beta_n a_n^{k-1}$, and solving for $d(a_j^k)$ yields

$$d(a_j^k) = \sum_{n=1}^{l} \beta_n a_n^{k-1} - d\left(\sum_{n=1}^{j-1} \frac{\alpha_n}{\alpha_j} a_n^k\right).$$

But this implies that

$$a_t^{k-1} = \sum_{n=1}^{l} \beta_n a_n^{k-1} - d\left(\sum_{n=1}^{j-1} \frac{\alpha_n}{\alpha_j} a_n^k\right),$$

contradicting the linear independence $\{a_j^{k-1}\}$. Thus, the canonical form of C_* is unique.

□

Proof of Theorem 2. Fix an arbitrary immersed GF-compatible filling of (Λ, f) with genus g with p immersed double points. Theorem 1 states that d_* can be chosen so that $H_{-k}(C(\Sigma_F, \{x_i\}), d_*) \cong GH^k(f)$. Recall that genus or an immersion point in the Lagrangian corresponds to two generators of $C(\Sigma_F, \{x_i\})$ whose indices are negatives of each other.

Suppose the generators of $C(\Sigma_F, \{x_i\})$ are ordered by their critical values. Let

$\widetilde{C_*}$ be the equivalent chain complex given by Proposition 12, such that for all a in $\widetilde{C_k}$, either $da = 0$ or $da = b$ for a unique generator $b \in \widetilde{C_{k-1}}$. This has the following interpretation: For each k, $\widetilde{C_k}$ has $\dim GH^k(f)$ generators of index k that get sent to 0 under d. The remaining generators makes up a set of

$$\mathcal{N}_F = 2p + 2g + 1 - \sum_k \dim GH^k(f)$$

elements. Let G_F denote this set. Since $d^2 = 0$, this produces a partition of G_F into pairs

$$G_F = \bigsqcup \{i_l^k, i_m^{k+1}\}$$

such that $d_{k+1}\left(i_m^{k+1}\right) = i_l^k$. Thus, $\mathcal{N}_F$ is a multiple of 2. But, for each pair $\left\{i_l^k, i_m^{k+1}\right\}$, there is a corresponding pair $\left\{i_{l'}^{-k-1}, i_{m'}^{-k}\right\}$. Thus, $\mathcal{N}_F$ is a multiple of 4 and we have

$$2p + 2g = \sum_k \dim GH^k(f) - 1 \mod 4, \text{ or}$$

$$p + g = \frac{1}{2}\left(\sum_k \dim GH^k(f) - 1\right) = \sum_{k=0}^{m} c_k \mod 2.$$

□

Theorem 2 justifies the lattice configurations of the diagrams in Figures 3 and 4. It is interesting to point out that for a given polynomial, if $c_0 = 0$, then the obstructed fillings all lie below a diagonal extending from the point with $p = g = 0$. If $c_0 \neq 0$, the the obstructed fillings all lie below a “check mark” whose left-most point is at $g = 0$ and $p = c_0$.

6.2 Long Exact Sequence of a Cobordism

Until this points, we considered cobordisms with $\Lambda_- = \emptyset$. We now consider the more general case and set out to prove Theorem 3. The following lemma which will help us identify terms of the long exact sequence in the proof of Theorem 3.

Lemma 10. *The following statements hold:*

1. $\left(\Delta^{\Omega}_{[v_-,t_-]}, \Delta^{-\mu}_{[v_-,t_-]}\right)$ *deformation retracts onto* $\left(\Delta^{\Omega}_{\{t_-\}}, \Delta^{-\mu}_{\{t_-\}}\right)$;
2. $\left(\Delta^{\Omega}_{[u_-,t_-]}, \Delta^{-\mu}_{[u_-,t_-]}\right)$ *deformation retracts onto* $\left(\Delta^{\Omega}_{\{t_-\}}, \Delta^{-\mu}_{\{t_-\}}\right)$;
3. $H^{k+N+1}\left(\Delta^{\Omega}_{\{t_-\}}, \Delta^{-\mu}_{\{t_-\}}\right) \cong \widetilde{GF}^k(f_-)$.

Proof. Statements 1 and 2 are direct applications of Lemma 7 by letting $a(t) = \lambda_\Omega(t)$ and $b(t) = \lambda_{-\mu}(t)$.

To prove 3, observe that since $\lambda_\Omega(t_-) > \overline{l_-}$,

$$\begin{aligned}\Delta^{\Omega}_{\{t_-\}} &= \{(t_-, x, \eta, \tilde{\eta}) : \Delta(t_-, x, \eta, \tilde{\eta}) < \Omega\} \\ &\cong \{(x, \eta, \tilde{\eta}) : \delta_-(x, \eta, \tilde{\eta}) < \lambda_\Omega(t_-)\} \\ &= \delta_-^{\omega}.\end{aligned}$$

Similarly,

$$\begin{aligned}\Delta^{-\mu}_{\{t_-\}} &= \{(t_-, x, \eta, \tilde{\eta}) : \Delta(t_-, x, \eta, \tilde{\eta}) < -\mu\} \\ &\cong \{(x, \eta, \tilde{\eta}) : \delta_-(x, \eta, \tilde{\eta}) < \lambda_{-\mu}(t_-)\},\end{aligned}$$

and, by Inequality 1 in Definition 9(i),

$$0 > \lambda_{-\mu}(t_-) = \frac{-\mu}{t_-} > -\underline{l_-}.$$

Thus, $\Delta^{-\mu}_{\{t_-\}}$ can be identified with $\delta^{-\epsilon}_-$.

□

Proof of Theorem 3. We will consider the triple of pairs $(X,Y) = (A,C) \cup (B,D)$ where:

$$(X,Y) = \left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right),$$

$$(A,C) = \left(\Delta^{\Omega}_{[v_-,t_-]}, \Delta^{-\mu}_{[v_-,t_-]}\right), \text{ and}$$

$$(B,D) = \left(\Delta^{\Omega}_{[u_-,u_+]}, \Delta^{-\mu}_{[u_-,u_+]}\right).$$

Note that

$$(A \cap B, C \cap D) = \left(\Delta^{\Omega}_{[u_-,t_-]}, \Delta^{-\mu}_{[u_-,t_-]}\right)$$

and consider the associated Mayer-Vietoris sequence:

$$\cdots \to H^{k+N+1}(X,Y) \to H^{k+N+1}(A,C) \oplus H^{k+N+1}(B,D) \to H^{k+N+1}(A \cap B, C \cap D) \to \cdots.$$

In the proof of Theorem 1, the isomorphism between $H^{k+N+1}\left(\Delta^{\Omega}_{[v_-,u_+]}, \Delta^{-\mu}_{[v_-,u_+]}\right)$ and $GF^k(f_+)$ is not dependent on the assumption that $\Lambda_- = \emptyset$. Thus we can identify $H^{k+N+1}(X,Y)$ with $GF^k(f_+)$ and $H^{k+N+1}(B,D)$ with $H_{n-k}\left(C(\Sigma, \{x_i\})\right)$, where n is the dimension of Σ. Lemma 10 implies that both $H^{k+N+1}(A, C)$ and $H^{k+N+1}(A \cap B, C \cap D)$ can be identified with $\widetilde{GF}^k(f_-)$. Thus we get the desired long exact sequence:

$$\cdots \to GH^k(f_+) \to \widetilde{GH}^k(f_-) \oplus H_{n-k}\left(C(\Sigma, \{x_i\})\right) \to \widetilde{GH}^k(f_-) \to \cdots .$$

□

To conclude this chapter, let us revisit Example 3 from the introduction. Using the long exact sequence in (4), it can be verified that $\dim \widetilde{GH}^2(f_-) = 0$. [*] Thus, the map $GH^3(f_+) \to \widetilde{GH}^3(f_-) \oplus H_{2-3}\left(C(\Sigma, \{x_i\})\right)$ is injective. Using the same long exact sequence in (4), it can also be verified that $\dim \widetilde{GH}^3(f_-) = 1$. [†] Since $\dim GH^3(f_+) = 3$, this means that $\dim H_{2-3}\left(C(\Sigma, \{x_i\})\right) = H^3\left(C(\Sigma, \{x_i\})\right) \geq 2$. Therefore, any immersed GF-compatible cobordism from (Λ_-, f_-) to (Λ_+, f_+) has at least two double points of index 3.

[*] Since $\dim H^2(\Lambda_-) = \dim H^3(\Lambda_-) = 0$, the map $GH^2(\Lambda_-) \to \widetilde{GH}^2(f_-)$ is an isomorphism. Since $\dim GH^2(\Lambda_-) = 0$, so does $\dim \widetilde{GH}^2(f_-)$.

[†] Since $\dim H^3(\Lambda_-) = \dim H^4(\Lambda_-) = 0$, the map $GH^3(\Lambda_-) \to \widetilde{GH}^3(f_-)$ is an isomorphism. Since $\dim GH^3(\Lambda_-) = 1$, so does $\dim \widetilde{GH}^3(f_-)$.

Chapter 7

Constructions

Now that we have a set of obstructions to the existence of immersed GF-compatible fillings, a natural next step is to see which immersed fillings are realizable. In this section, we first show that an immersed Lagrangian cobordism can be obtained from a homotopy of generating families. We also construct a series of combinatorial moves that can be performed on front diagrams that guarantee the existence of an immersed gf-cobordism. With these moves, we prove a partial converse to Corollary 1 by showing that for any polynomial satisfying one-dimensional duality, there exists a pair (Λ, f) having that polynomial with a minimal immersed filling. Furthermore, we give an algorithm for creating new immersed fillings from existing ones with additional immersion points or genus. Together, these show existence of the entire upper diagonal region of the lattice diagram for a polynomial, as in Figure 7.

7.1 Generating Family Homotopies

Lemma 11. *Suppose* $f_s : M \times \mathbb{R}^N \to \mathbb{R}$ *with* $s \in \mathbb{R}$ *is a homotopy of generating families such that for each* s*,* f_s *generates a Legendrian* Λ_s*. Suppose further that there exists* $s_0 < s_1$ *such that for all* $s < s_0$*,* $f_s = f_{s_0}$ *and for all* $s > s_1$*,* $f_s = f_{s_1}$*. Then, letting* $t = e^s$*, there is an immersed* gf*-compatible Lagrangian cobordism* $\mathcal{L}$ *from* Λ_{s_0} *to* Λ_{s_1} *generated by* $F : \mathbb{R}_+ \times M \times \mathbb{R}^N \to \mathbb{R}$*, where*

$$F(t, x, \eta) = t f_t(x, \eta).$$

Furthermore, if $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ *is a critical point of* Δ*, then* $(x_0, \eta_0, \widetilde{\eta_0})$ *is a critical point of* δ_{t_0} *with critical value equal to*

$$t_0 \frac{\partial f_t}{\partial t}(x_0, \eta_0) - t_0 \frac{\partial f_t}{\partial t}(x_0, \widetilde{\eta_0}).$$

The indices of these critical points satisfy the following:

$$\mathrm{Ind}_{(t_0, x_0, \eta_0, \widetilde{\eta_0})} \Delta = \mathrm{Ind}_{(x_0, \eta_0, \widetilde{\eta_0})} \delta_{t_0} + \begin{cases} 0, & t\frac{\partial^2 \delta_t}{\partial t^2} + 2\frac{\partial \delta_t}{\partial t} > 0 \\ 1, & t\frac{\partial^2 \delta_t}{\partial t^2} + 2\frac{\partial \delta_t}{\partial t} < 0 \end{cases}$$

Proof. Since f_s generates a Legendrian γ_s for all s, 0 must be a regular value of $\frac{\partial f_s}{\partial \eta}(x, \eta)$, and hence a regular value of $\frac{\partial f_t}{\partial \eta}(x, \eta)$ for all t. For all elements of $\frac{\partial f_t}{\partial \eta}(x, \eta)^{-1}(0)$, the derivative map is the $N \times (N + m)$ matrix:

$$d\left(\frac{\partial f_t}{\partial \eta}(x, \eta)\right) = \left(\frac{\partial^2 f_t}{\partial \eta \partial x}, \frac{\partial^2 f_t}{\partial \eta^2}\right),$$

which has rank N. Since $t \neq 0$ and $F = tf_t$, elements of $\frac{\partial F}{\partial \eta}(t,x,\eta)^{-1}(0)$ are in correspondence with elements of $\frac{\partial f_t}{\partial \eta}(x,\eta)^{-1}(0)$, and the derivative map is the $N \times (N+m+1)$ matrix:

$$\begin{aligned} d\left(\frac{\partial F}{\partial \eta}(t,x,\eta)\right) &= \left(\frac{\partial^2 F}{\partial \eta \partial t}, \frac{\partial^2 F}{\partial \eta \partial x}, \frac{\partial^2 F}{\partial \eta^2}\right) \\ &= \left(\frac{\partial^2 t f_t}{\partial \eta \partial t}, \frac{\partial^2 t f_t}{\partial \eta \partial x}, \frac{\partial^2 t f_t}{\partial \eta^2}\right) \\ &= \left(\frac{\partial f_t}{\partial \eta}, \frac{t\partial^2 f_t}{\partial \eta \partial x}, \frac{t\partial^2 f_t}{\partial \eta^2}\right), \end{aligned}$$

which also has rank N. Thus, 0 is also be a regular value of $\frac{\partial F}{\partial \eta}$. By Lemma 3.7 in [4], this implies that F generates an immersed Lagrangian.

The double points of this immersed Lagrangian occur when there exists $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ such that

$$\begin{aligned} &\left(t_0, x_0, f_{t_0}(x_0,\eta_0) + t_0 \frac{\partial f_{t_0}}{\partial t}(x_0,\eta_0), t_0 \frac{\partial f_{t_0}}{\partial x}(x_0,\eta_0)\right) \\ = &\left(t_0, x_0, f_{t_0}(x_0,\widetilde{\eta_0}) + t_0 \frac{\partial f_{t_0}}{\partial t}(x_0,\widetilde{\eta_0}), t_0 \frac{\partial f_{t_0}}{\partial x}(x_0,\widetilde{\eta_0})\right). \end{aligned} \tag{20}$$

Recall that $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ is also a critical point of Δ. A direct calculation of partial derivatives shows that $(x_0, \eta_0, \widetilde{\eta_0})$ is also a critical point of δ_{t_0}. Equality of the third coordinates in (20) gives us the following equation:

$$f_{t_0}(x_0,\widetilde{\eta_0}) - f_{t_0}(x_0,\eta_0) = t_0 \frac{\partial f_{t_0}}{\partial t_0}(x_0,\eta_0) - t_0 \frac{\partial f_{t_0}}{\partial t_0}(x_0,\widetilde{\eta_0}). \tag{21}$$

Notice that the left hand side of this equation is $\delta_{t_0}(x_0, \eta_0, \widetilde{\eta_0})$, making the right hand side the critical value of the critical point $(x_0, \eta_0, \widetilde{\eta_0})$ of δ_{t_0}.

It remains to show the correspondence between the indices of critical points. The Hessian matrix of Δ takes the following form:

$$\left(\begin{array}{c|ccc}
t\dfrac{\partial^2\delta_t}{\partial t^2} + 2\dfrac{\partial\delta_t}{\partial t} & t\dfrac{\partial^2\delta_t}{\partial t\partial x} + \dfrac{\partial\delta_t}{\partial x} & t\dfrac{\partial^2\delta_t}{\partial t\partial \eta} + \dfrac{\partial\delta_t}{\partial \eta} & t\dfrac{\partial^2\delta_t}{\partial t\partial \tilde{\eta}} + \dfrac{\partial\delta_t}{\partial t\tilde{\eta}} \\
\hline
t\dfrac{\partial^2\delta_t}{\partial t\partial x} + \dfrac{\partial\delta_t}{\partial x} & & & \\
t\dfrac{\partial^2\delta_t}{\partial t\partial \eta} + \dfrac{\partial\delta_t}{\partial \eta} & & t_0 H(\delta_{t_0}) & \\
t\dfrac{\partial^2\delta_t}{\partial t\partial \tilde{\eta}} + \dfrac{\partial\delta_t}{\partial t\tilde{\eta}} & & &
\end{array}\right)$$

By the Morse Lemma, since $(x_0, \eta_0, \widetilde{\eta_0})$ is a critical point of δ_{t_0}, there exist local coordinates in a neighborhood of $(x_0, \eta_0, \widetilde{\eta_0})$ such that

$$\delta_{t_0}(x, \eta, \tilde{\eta}) = \delta_{t_0}(x_0, \eta_0, \widetilde{\eta_0}) + Q_0(x) + Q_1(\eta) + Q_2(\tilde{\eta})$$

where each Q_i is a quadratic function of the given variable. Thus, we get the following equalities:

$$
\begin{aligned}
0 &= \frac{\partial \delta_{t_0}}{\partial t}(x_0, \eta_0, \widetilde{\eta_0}); \\
0 &= \frac{\partial \delta_{t_0}}{\partial x}(x_0, \eta_0, \widetilde{\eta_0}) = Q_0'(x_0); \\
0 &= \frac{\partial \delta_{t_0}}{\partial \eta}(x_0, \eta_0, \widetilde{\eta_0}) = Q_1'(\eta_0); \\
0 &= \frac{\partial \delta_{t_0}}{\partial \tilde{\eta}}(x_0, \eta_0, \widetilde{\eta_0}) = Q_2'(\widetilde{\eta_0}),
\end{aligned}
$$

and the Hessian matrix takes the following form:

$$
\left(\begin{array}{c|ccc}
t\frac{\partial^2 \delta_t}{\partial t^2} + 2\frac{\partial \delta_t}{\partial t} & 0 & 0 & 0 \\
\hline
0 & & & \\
0 & & t_0 H(\delta_{t_0}) & \\
0 & & &
\end{array}\right)
$$

By the symmetry of the matrix and the well placed zeros, the lower block of the matrix can be diagonalized and the eigenvalues can be picked off the diagonal. The number of negative eigenvaules is exactly $\mathrm{Ind}_{(x_0,\eta_0,\widetilde{\eta_0})} t_0\delta_{t_0}$. Thus we obtain the desired equality.

□

Remark 16. For F defined as in Lemma 11, there is a one-to-one correspondence between double points of the Lagrangian generated by F and Reeb chords of Λ_t with height $\left| t\frac{\partial f_t}{\partial t}(x,\eta) - t\frac{\partial f_t}{\partial t}(x,\tilde{\eta}) \right|$. In the next section, we produce generating family homotopies using a series of combinatorial moves with the clasping and unclasping moves in Figure 23 corresponding to an immersed double point in the Lagrangian. It is important to note that the immersed double point in the Lagrangian does not occur exactly when the two strands in a clasp meet, but rather when they pass a Reeb chord of height $\left|t\frac{\partial f_t}{\partial t}(x,\eta) - t\frac{\partial f_t}{\partial t}(x,\tilde{\eta})\right|$.

In the case that f_t is generating family homotopy such that each f_t generates a Legendrian knot Λ_t, Lemma 11 gives us an easy way to determine the indices of immersion points. Suppose for some t_0, Λ_{t_0} has a Reeb chord c of height $\left|t_0\frac{\partial f_{t_0}}{\partial t}(x_0,\eta_0) - t_0\frac{\partial f_{t_0}}{\partial t}(x_0,\tilde{\eta}_0)\right|$, where $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ is the corresponding critical point of Δ. Let b denote the branch containing the point, $(x_0, \partial_x f(x_0,\eta_0), f(x_0,\eta_0))$ and $\tilde{b}$ denote the branch containing the point, $(x_0, \partial_x f(x_0,\widetilde{\eta_0}), f(x_0,\widetilde{\eta_0}))$. In neighborhoods U and $\tilde{U}$ of (x_0,η_0) and $(x_0,\widetilde{\eta_0})$, respectively, there exist local coordinates such that

$$f_{t_0}(x,\eta) = g(x) + Q(\eta)$$

and

$$f_{t_0}(x,\tilde{\eta}) = \tilde{g}(x) + \tilde{Q}(\tilde{\eta}).$$

Since $(x_0, \eta_0, \widetilde{\eta_0})$ is a non-degenerate critical point of δ_f, x_0 is a non-degenerate critical point of $\tilde{g}-g$. Furthermore, g and $\tilde{g}$ locally trace the front-projections of the branches. That is,

$$\pi_{xz}(b) = \{(x, g(x) + Q(\eta_0))\} \qquad \text{and} \qquad \pi_{xz}(\tilde{b}) = \left\{(x, \tilde{g}(x) + \tilde{Q}(\widetilde{\eta_0}))\right\}.$$

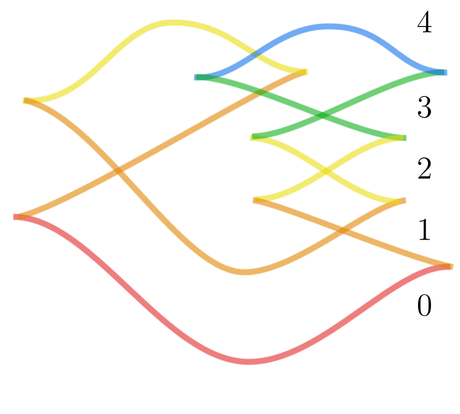


Figure 20: Example assignment of Maslov potentials to the spanning arcs of this front projection of a Legendrian $m(5_2)$ knot.

Definition 14. Suppose Λ is a Legendrian knot in J^1M with a Reeb chord c corresponding to a critical point $(x_0, \eta_0, \widetilde{\eta_0})$ of δ. For g and $\tilde{g}$ defined as above, define the **graph index**, $G(x_0)$ to be the Morse index of $\tilde{g} - g$ at x_0.

Definition 15. Let Λ be a Legendrian knot with rotation number 0 and consider its front projection. Define a **spanning arc** to be a path between two cusps on the front diagram. To each spanning arc, the **Maslov potential** μ assigns an integer in such a way that at any cusp, the integer assigned to the upper arc is one more than the integer assigned to the lower arc.

Remark 17. By our construction, η_0 and $\widetilde{\eta_0}$ are also non-degenerate critical points of Q and $\tilde{Q}$, respectively. The indices of these critical points are given by the Maslov potential. That is, $\text{Ind}_{\eta_0}(Q(\eta)) = \mu(b)$ and $\text{Ind}_{\widetilde{\eta_0}}(\widetilde{Q}(\tilde{\eta})) = \mu(\tilde{b})$.

By reparamaterizing if necessary, the homotopy f can be assumed to be approximately linear in the t coordinate, we can obtain the index of a critical point of Δ

from a shift of the difference in branch indices of the corresponding Reeb chords. To that end, let $\epsilon > 0$ be chosen such that $\epsilon \ll |\delta_{t_0}(x_0, \eta_0, \widetilde{\eta_0})|$.

Corollary 3. *Suppose γ_t is a generating family homotopy satisfying $\left|\frac{\partial^2}{\partial t^2}\delta_t\right| < \epsilon$ that induces a Lagrangian generated by F. If $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ is a critical point of Δ_F then the index of $p = (t_0, x_0, \eta_0, \widetilde{\eta_0})$ satisfies the following:*

$$\mathrm{Ind}_{(t_0,x_0,\eta_0,\widetilde{\eta_0})}\Delta_F = \mu(\tilde{\eta}) - \mu(\eta) + N + 1 + \begin{cases} G(x_0), & \text{if } \Delta(p) > 0 \\ -G(x_0), & \text{if } \Delta(p) < 0. \end{cases} \tag{22}$$

Proof. Let f be a generating family homotopy and F be defined as in Lemma 11. Suppose $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ is a critical point of Δ_F. We will first compute the index of $(x_0, \eta_0, \widetilde{\eta_0})$ as a critical point of δ_{t_0}.

There exist local coordinates in neighborhoods of (x_0, η_0) and $(x_0, \widetilde{\eta_0})$ such that

$$f_{t_0}(x, \eta) = g(x) + Q(\eta)$$

and

$$f_{t_0}(x, \tilde{\eta}) = \tilde{g}(x) + \widetilde{Q}(\tilde{\eta}).$$

Thus,

$$\begin{aligned}
\mathrm{Ind}_{(x_0,\eta_0,\widetilde{\eta_0})}\delta_{t_0} &= \mathrm{Ind}_{(x_0,\eta_0,\widetilde{\eta_0})}\left(f(x,\tilde{\eta}) - f(x,\eta)\right)\\
&= \mathrm{Ind}_{(x_0,\eta_0,\widetilde{\eta_0})}\left(\tilde{g}(x) + \widetilde{Q}(\tilde{\eta}) - (g(x) + Q(\eta))\right)\\
&= \mathrm{Ind}_{x_0}\left(\tilde{g}(x) - g(x)) + \mathrm{Ind}_{\widetilde{\eta_0}}(\widetilde{Q}(\tilde{\eta})) + \mathrm{Ind}_{\eta_0}(-Q(\eta)\right)\\
&= G(x_0) + \mathrm{Ind}_{\widetilde{\eta_0}}(\widetilde{Q}(\tilde{\eta})) + (N - \mathrm{Ind}_{\eta_0}(Q(\eta)))\\
&= G(x_0) + \mu(\tilde{\eta}) - \mu(\eta) + N.
\end{aligned}$$

It remains to analyze the contribution to the index of Δ_F from the t coordinate. Since $|\frac{\partial^2}{\partial t^2}\delta_t| < \epsilon$, we need only determine the sign of $\frac{\partial}{\partial t}\delta_t$. Solving for $\dfrac{\partial f_{t_0}}{\partial t_0}(x_0,\widetilde{\eta_0}) - \dfrac{\partial f_{t_0}}{\partial t_0}(x_0,\eta_0)$ in (21), we get

$$\frac{\partial}{\partial t}\delta_{t_0}(x_0,\eta_0,\widetilde{\eta_0}) = \frac{-1}{t}\delta_{t_0}(x_0,\eta_0,\widetilde{\eta_0}).$$

Thus, if the critical value, $\delta_{t_0}(x_0,\eta_0,\widetilde{\eta_0}) > 0$, then $\frac{\partial}{\partial t}\delta_t < 0$ and by Lemma 11, the index shifts by 1, yielding the $+1$ in (22). If $\delta_{t_0}(x_0,\eta_0,\widetilde{\eta_0}) < 0$, then $\frac{\partial}{\partial t}\delta_t > 0$, and the index shifts by 0, which yields the term $1 - G(x_0)$ in (22).

□

Remark 18. For critical points $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ and $(t_0, x_0, \widetilde{\eta_0}, \eta_0)$,

$$\mathrm{Ind}_{(t_0,x_0,\widetilde{\eta_0},\eta_0)}\Delta_F - (N+1) = -\left(\mathrm{Ind}_{(t_0,x_0,\widetilde{\eta_0},\eta_0)}\Delta_F - (N+1)\right).$$

Definition 16. Let d be a double point corresponding to critical points $(t_0, x_0, \eta_0, \widetilde{\eta_0})$ and $(t_0, x_0, \widetilde{\eta_0}, \eta_0)$. Define the **index of the double point** to be

$$\operatorname{Ind}(d) = \left|\operatorname{Ind}_{(t_0,x_0,\widetilde{\eta_0},\eta_0)}\Delta_F - (N+1)\right|.$$

7.2 Constructions from Front Diagrams

It is known that if two front diagrams are related by a series of Legendrian Reidemeister moves (Figure 21) and surgeries (Figure 22), then there exists an embedded GF-compatible cobordism between them. See, for example [4]. We shall introduce additional "clasping" and "unclasping" moves (Figure 23) that will guarantee the existence of an immersed GF-compatible cobordism. In order to guarantee generating family compatibility of the cobordism, certain conditions will be specified for the surgery and (un)clasping moves.

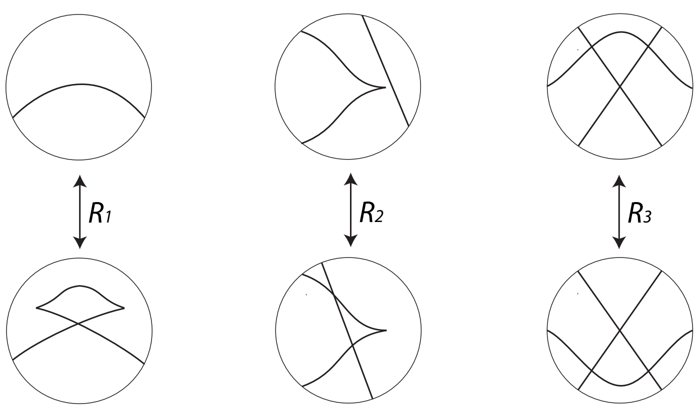


Figure 21: Legendrian Reidemeister Moves. There is also a version of the R_1 move with the diagram flipped about a horizontal axis. There are versions of the R_2 move with the diagram flipped along either the vertical or horizontal axes.

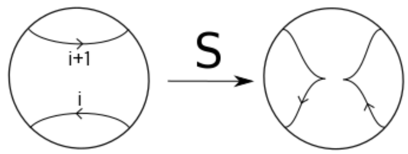


Figure 22: A surgery move replaces a 0-tangle with an ∞-tangle. It can be performed on strands with opposite orientation in the same "eye" of a graded normal ruling. Two surgeries result in additional genus in the cobordism.

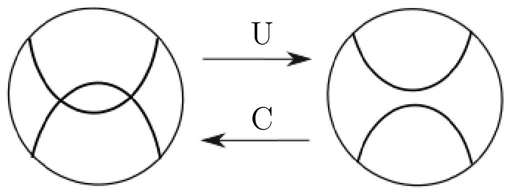


Figure 23: An unclasping move replaces clasped strands with a 0-tangle. A clasping move replaces a 0-tangle with a clasp.

In order to define the conditions under which the surgery and (un)clasping moves are admissible, we must first recall the notion of normal rulings, an idea introduced by people such as Eliashberg [15], Chekanov-Pushkar [6], and Fuchs [17]. Roughly, the idea is to resolve crossings in the front diagram in order to decompose it into "eyes", or unknots with $tb = -1$ and no crossings.

Definition 17. Given a front diagram of a Legendrian knot Λ such that $r(\Lambda) = 0$, an **eye** consists of:

1. A left and a right cusp, and
2. Two disjoint paths joining these cusps that meet only at their shared endpoints.

A **ruling** of a front diagram is a decomposition of the diagram into eyes such that paths of different eyes meet only at crossings. A **switch** in a ruling is a crossing in the original diagram where two paths of the ruling meet but don't cross. A ruling is **normal** if the switches are all of the type in Figure 24(a). A normal ruling is **graded** if at each switch, the difference in Maslov potentials of the strands is 0.

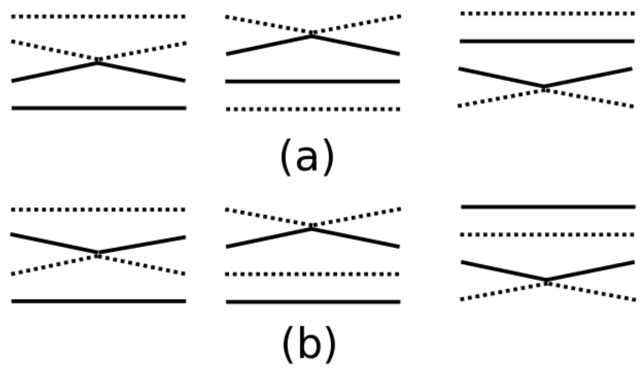


Figure 24: The switches in (a) admit normal rulings. Paired strands must be either disjoint from or nested with other pairs. The switches in (b) do not admit normal rulings as the two pairs overlap but are not nested.

The following proposition due to Chekanov-Pushkar [6] gives us an important relationship between graded normal rulings and generating families. Fuchs-Rutherford also gave an explicit construction of a generating family from a normal ruling in [18].

Proposition 13 (Section 12 in [6], Theorem 2.4 in [18])**.** *A Legendrian knot with rotation number* 0 *has a linear-at-infinity generating family if and only if its front diagram has a graded normal ruling.*

This proposition allows us to describe the conditions under which the surgery and (un)clasping moves are generating family compatible purely from front diagrams. The

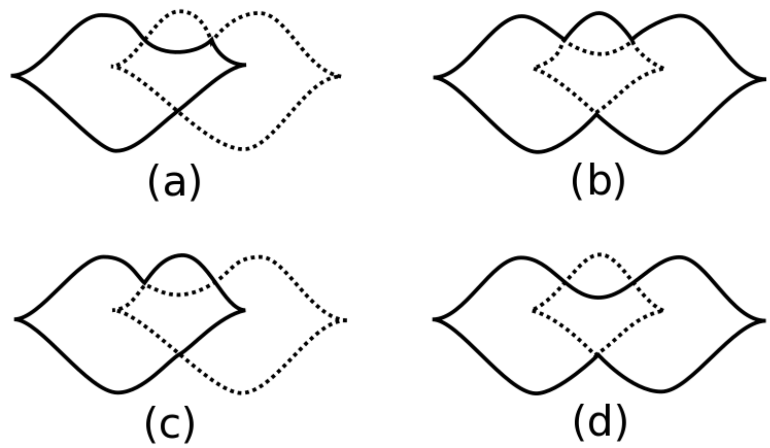


Figure 25: Four rulings of a Legendrian trefoil. The rulings in (a)-(c) are graded normal rulings and the ruling in (d) is not since the switch is of the last type in Figure 24(b).

surgery moved can be applied to pairs of strands that correspond to the same eye in the normal ruling, a proof for which can be found in [4]. The conditions under which an (un)clasping move is generating family compatible are described in the following theorem.

Theorem 14. *Suppose* Λ_0 *and* Λ_1 *are two front diagrams with graded normal rulings and* Λ_1 *can be obtained from* Λ_0 *via a clasping or unclasping move c such that either*

- *neither of the crossings in the clasp are switches, or*
- *both of the crossings in the clasp are switches,*

as in Figure 26. Then there exist generating families for Λ_0 *and* Λ_1 *and an immersed GF-compatible cobordism between them. Furthermore, the Lagrangian has one im-*

mersed double point and the corresponding critical points of Δ *have index*

$$\pm\left(\mu(\tilde{\eta})-\mu(\eta)\right)+N+1\pm\begin{cases}0, & \text{if } c \text{ is an unclasp,}\\ 1, & \text{if } c \text{ is a clasp.}\end{cases} \tag{23}$$

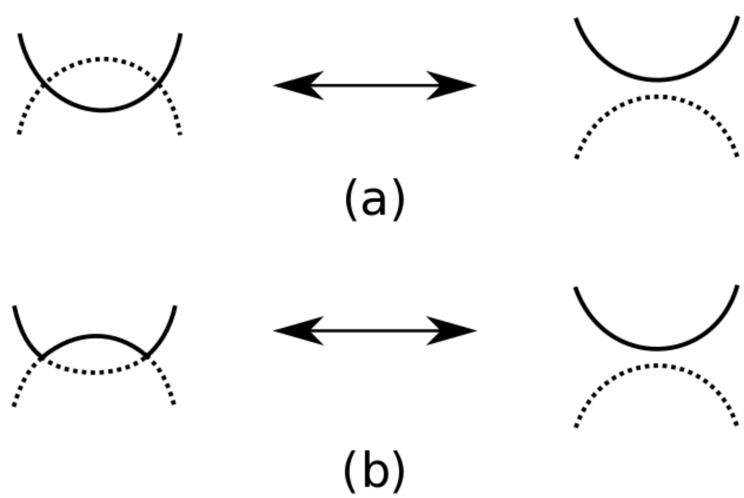


Figure 26: Two cases for a (un)clasp. In (a), there are no switches at the clasp. In (b), switches occur at both crossings in that clasp.

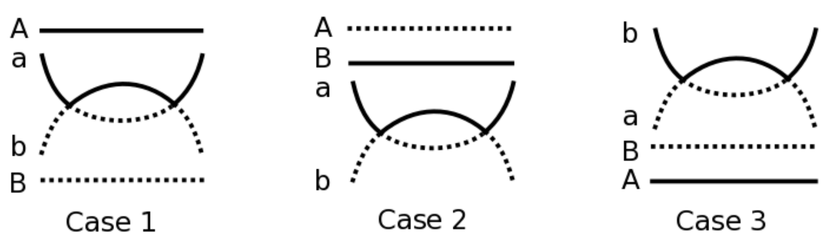


Figure 27: Three subcases for a clasp with two switches.

The proof of this theorem relies on Fuchs-Rutherford's construction in Section 3 of [18] of a generating family from a graded normal ruling by means of a Morse complex sequence. While we have omitted background on Morse complex sequences in this paper, the reader is encouraged to refer to [21], [22], and [23] for a thorough discussion on Morse complex sequences and their connection to generating families. In the proof of Theorem 14, we make use of the fact that if a front diagram can be given a Morse complex sequence, then the Legendrian has a generating family, i.e. the following:

Proposition 15 (Section 12 in [6], Proposition 4.6 in [22])**.** *A Legendrian knot with rotation number* 0 *has a linear-at-infinity generating family if and only if its front diagram can be given a Morse complex sequence.*

Proof of Theorem 14. Let Λ_0 be a front diagram with a graded normal ruling. Following Fuchs-Rutherford's construction in Section 3 of [18], assume that the Maslov potential of all strands in Λ_0 is greater than 2 and less than $N-1$. We will construct a family of functions $f_s : M \times \mathbb{R}^N \to \mathbb{R}$ with $s \in [0,1]$ in terms of the Morse complex sequence. For fixed s, let $f_{s,x} = f_s(x, \cdot)$. For values of s away from switches, $f_{s,x}$ takes the following simple form: for each "eye" in the normal ruling, we get two critical points whose indices agree with the Maslov potentials of the strands, as well as a single trajectory between them. Anywhere a switch occurs in the normal ruling, a handle slide is performed immediately before any after the crossing.

In the case that the clasp has no switches, we directly apply this method to construct a generating family for the Legendrian recorded by its Morse complex sequence.

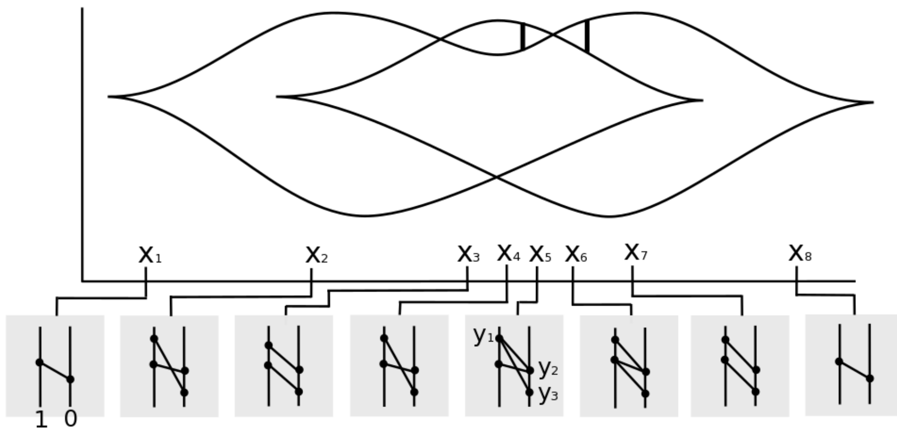


Figure 28: This Morse complex sequence corresponds to the generating family of the Legendrian trefoil pictured in Figure 13 and to the graded normal ruling pictured in Figure 25a. A Morse complex sequence can be thought of as a 1-parameter family of chain complexes which encodes the key data of the generating family. Each grey box has a chain complex representing the generating family restricted to the specified value of x. The points represent critical points and the vertical lines are ordered according to their index (0 or 1 in this example). The heights of the points represent their critical values. The sloped lines indicate that there is a trajectory between the two critical points. For example, $\partial y_1 = y_2 + y_3$. The vertical marks on the Legendrian knot represent handleslides, which are trajectories between two critical points of the same index.

We then extend this to a 1-parameter family of Morse complex sequences, which gives us a homotopy of generating families. Such a series of Morse complex sequences is depicted in Figure 29. Applying Lemma 11, this induces a GF-compatible Lagrangian cobordism.

In the case that the Legendrian has a ruling such that the clasp in question has two switches, there are three cases for the relative heights of paired strands in the ruling. These are depicted in Figure 27. One-parameter families of Morse complex sequences for Cases 1 and 2 are given in Figure 30. Notice that handle slides occur immediately before and after the crossings of the clasp. In order to construct a 1-parameter family of Morse complex sequences, the resulting unclasp also has two handleslides. Case 3 is completely analogous to Case 2. This proves the existence of a generating family homotopy f_s from f_0 to f_1, where f_0 generates Λ_0 and f_1 generates Λ_1. To illustration what this homotopy of generating families might look like, Figure 31 depicts possible level sets for such a generating family homotopy. Again applying Lemma 11, we get a GF-compatible Lagrangian cobordism.

It remains to show that performing this move corresponds to an immersed double point in the Lagrangian. Our strategy is to define local coordinates, as described earlier in this section, on the strands of the clasp and apply Corollary 3. Define a linear function $h(s) = -ms + b$ with $m, b > 0$ and $m < b$. For $s \in [0, 1]$, a homotopy of generating families describing the unclasping move can be locally given by:

$$f_s(x, \eta) = x^2 + h(s) + Q(\eta)$$

$$\tilde{f}_s(x, \tilde{\eta}) = -x^2 + Q(\tilde{\eta})$$

Letting $t = e^s$, locally we have $\delta_t(x, \eta, \tilde{\eta}) = -2x^2 - h(\ln(t)) + Q(\tilde{\eta}) - Q(\eta)$. Thus, the signed Reeb chord heights are locally given by $-h(s)$. By Lemma 11, it suffices to show that there exists s such that Λ_s has a Reeb chord with height equal to

$$t\frac{\partial f_t}{\partial t}(x, \eta) - t\frac{\partial f_t}{\partial t}(x, \tilde{\eta}) = t\frac{h'(\ln(t))}{t} = h'(\ln(t)) = -m$$

Using this local model, Λ_s has a Reeb chord of signed height $-b$ at $s = 0$, and Reeb chord of signed height $b - m$ at $s = 1$. Thus, by choice of b and m, there exists $s \in [0, 1]$ such that Λ_s has a Reeb chord with signed height equal to $-m$.

To calculate the indices of the critical points of Δ associated to this immersed point, we study $G(x_0)$. Suppose $(t_0, x_0, \eta_0, \tilde{\eta}_0)$ and $(t_0, x_0, \tilde{\eta}_0, \eta_0)$ are the associated critical points. Locally, $\tilde{g}(x), g(x) = \pm x^2$. The sign is determined based on the strands that b and $\tilde{b}$ correspond to in the clasp. It should be noted that, in practice, we perform these moves for s *decreasing*.

For an unclasping move with a positive critical value, $\tilde{b}$ corresponds to the strand touching the positive end of the Reeb chord. So locally, $\tilde{g}(x) = x^2$ and $g(x) = -x^2$, which by definition of the graph index, implies that $G(x_0) = 0$. If the critical value is negative, then $\tilde{b}$ corresponds to the strand touching the positive end of the Reeb chord and locally, $\tilde{g}(x) = -x^2$ and $g(x) = x^2$. This implies that $G(x_0) = 1$. The opposite phenomena occurs for a clasping move. If the critical value is positive, $\tilde{b}$ corresponds to the strand touching the positive end of the Reeb chord and locally, $\tilde{g}(x) = -x^2$ and $g(x) = x^2$, which implies $G(x_0) = 1$. If the critical value is negative, $G(x_0) = 0$.

□

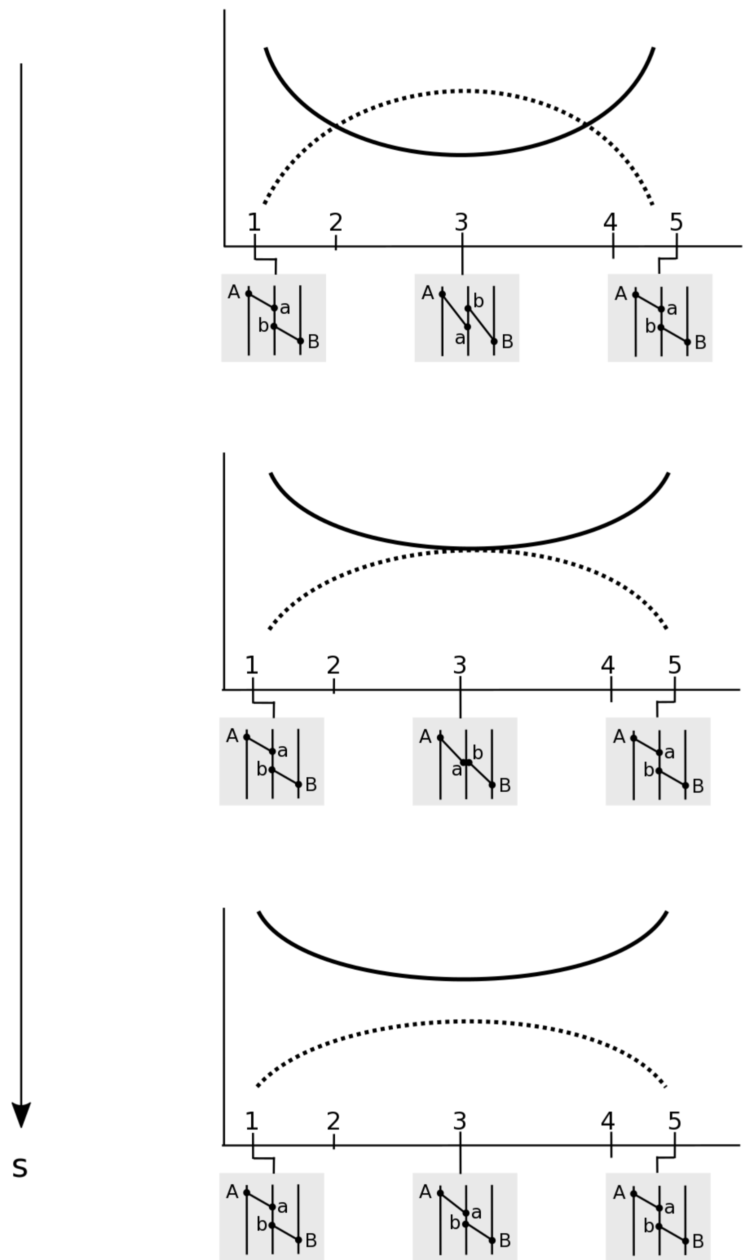


Figure 29: A one-parameter family of Morse complex sequences, which induces a generating family homotopy, for an (un)clasp move of the type in Figure 26(a).

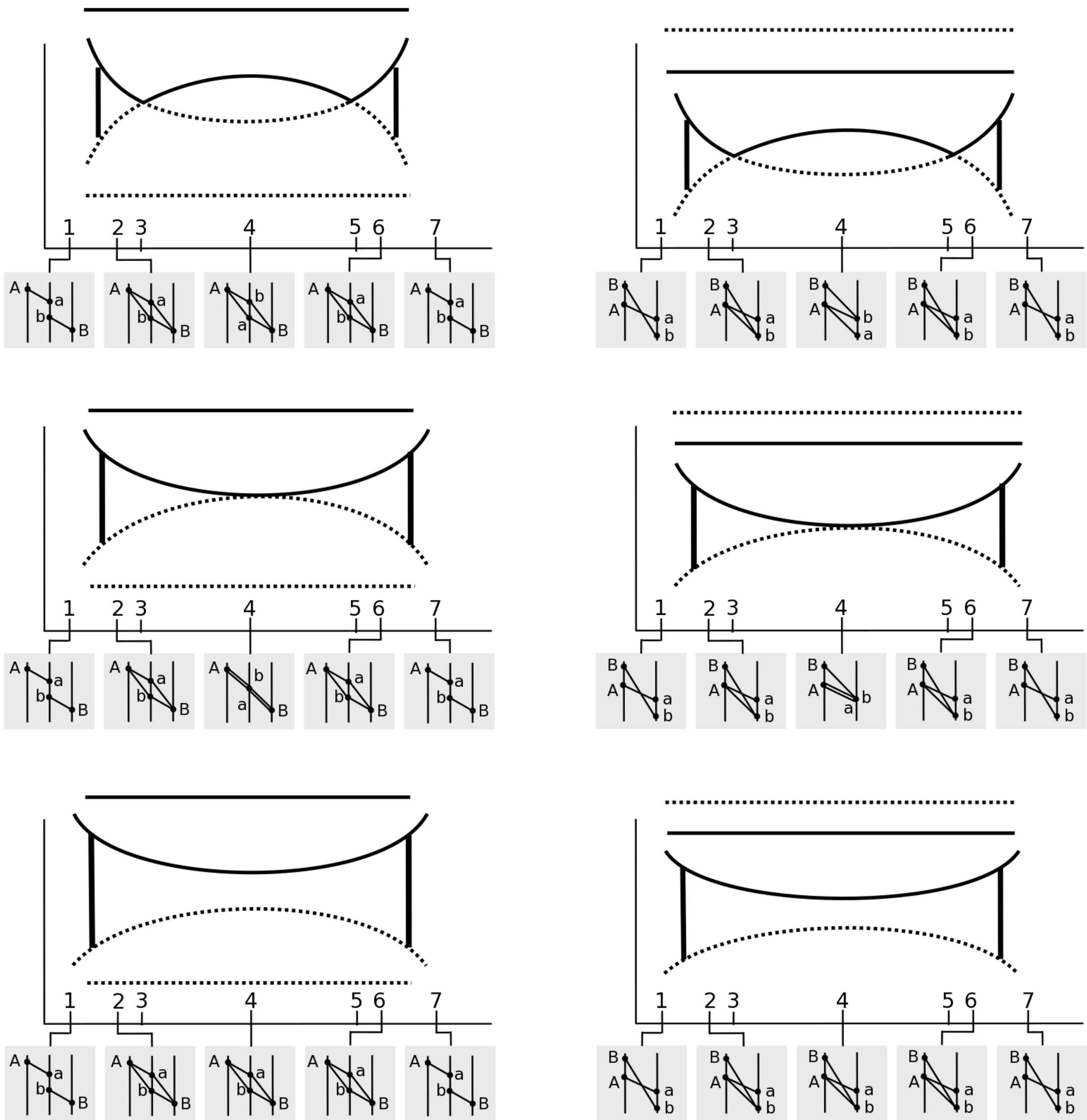


Figure 30: (Left) A one-parameter family of Morse complex sequences, which induces a generating family homotopy, for an (un)clasp move in Case 1 of Figure 27. (Right) A one-parameter family of Morse complex sequences for an (un)clasp in Case 2.

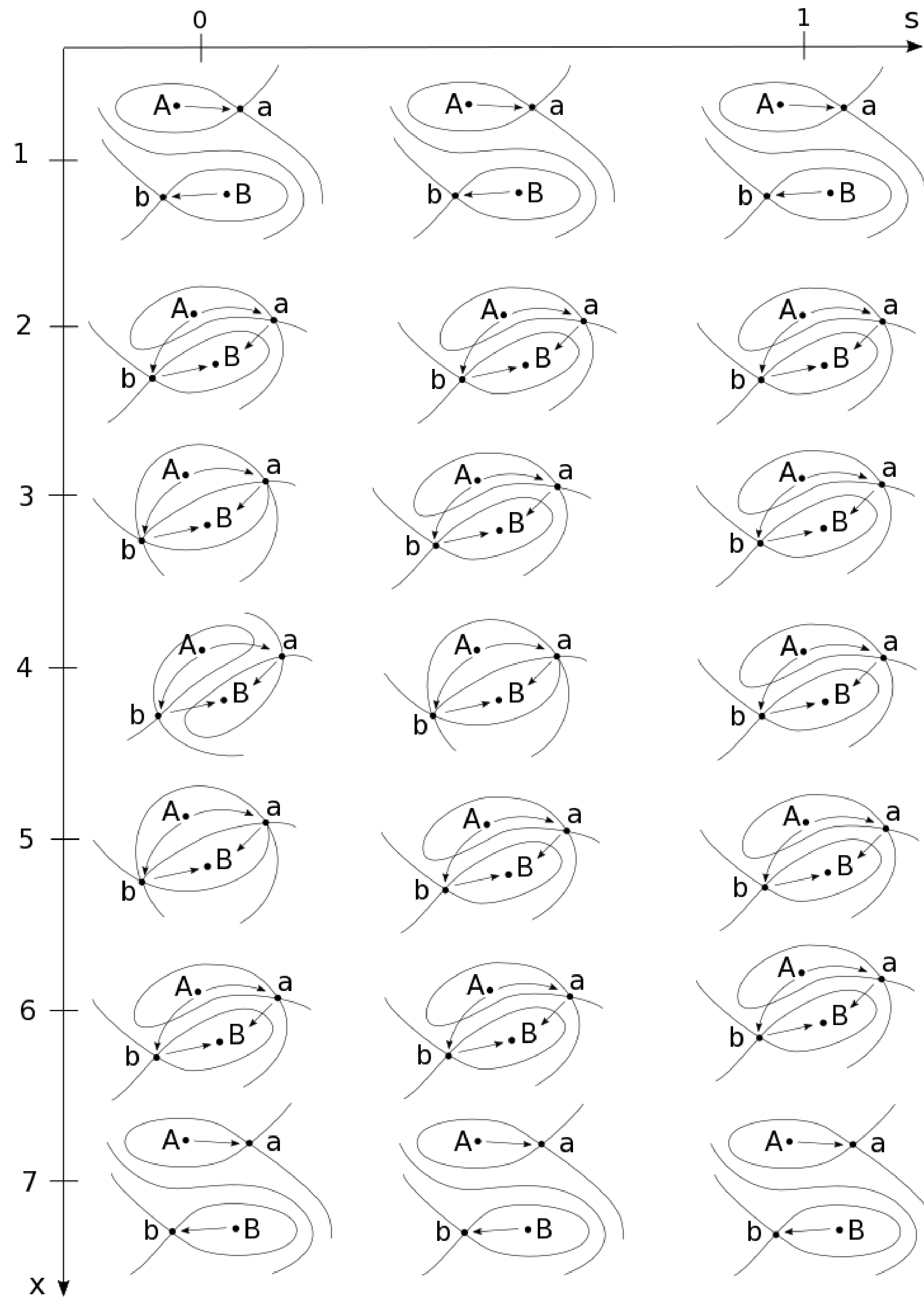


Figure 31: Possible level sets for the generating family homotopy for a Case 1 (un)clasp with two switches.

7.3 Realizing Immersed Fillings and Cobordisms

In this section, we apply the moves constructed above to some concrete examples. First, consider the Legendrian knot $\Lambda_+ = m(5_2)$ that has polynomial $t^{-2}+t+t^2$ from Figure 2b and Example 1. The set of moves in Figure 32 below shows that there exists a generating family f_+ for Λ_+ and an immersed GF-compatible cobordism from (Λ_+, f_+) to the $(-1,0)$-unknot, that is the unknot with $tb = -1$ and $r = 0$. Since one unclasping move was performed, the resulting cobordism has one immersed double point of index 2. Furthermore, since the $(-1,0)$-unknot has an embedded GF-compatible disk filling, this construction gives an immesred GF-compatible *filling* of (Λ_+, f_+) with one immersed double point of index 2.

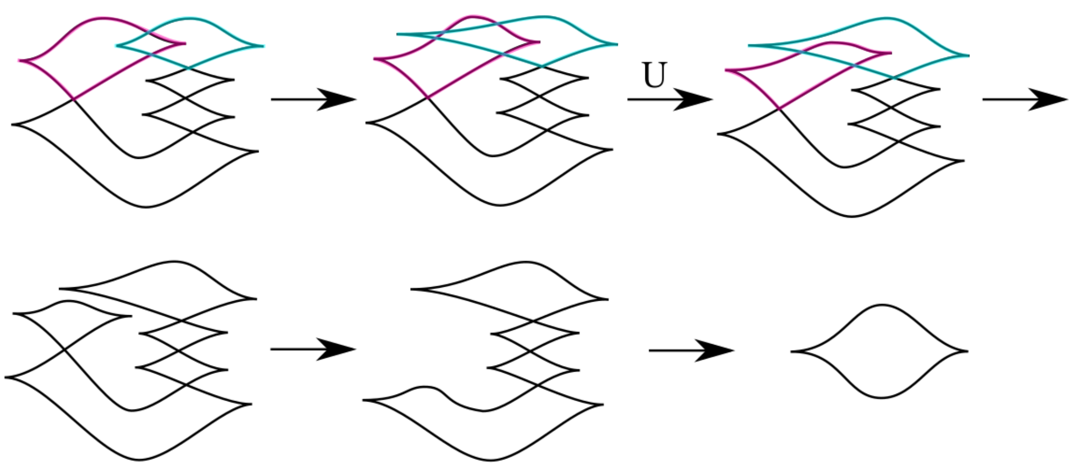


Figure 32: This Legendrian $m(5_2)$ knot can be related to the Legendrian unknot with $tb = -1$ by performing one unclasping move along with a series of Legendrian Reidemeister moves. This implies the existence of an immersed GF-compatible filling with one double point.

Next, consider the Legendrian knots Λ_- and Λ_+ which are, respectively, the $m(6_1)$

knot with polynomial $t^{-3}+t+t^3$ and $m(10_1)$ knot with polynomial $3t^{-3}+t+3t^3$ from Figure 5 and Example 3. Performing two unclasping moves on Λ_+ as in Figure 33 implies that there exists generating families f_- for Λ_- and f_+ for Λ_+ and an immersed GF-compatible cobordism (L, F) from (Λ_-, f_-) to (Λ_+, f_+) with two immersed double points of index 3.

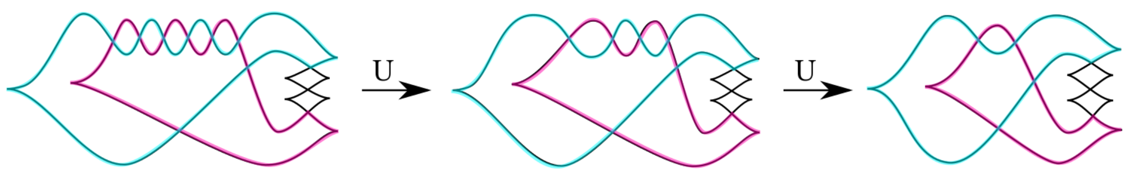


Figure 33: This Legendrian knot $\Lambda_+ = m(10_1)$ can be related to the Legendrian $\Lambda_- = m(6_1)$ knot by performing two unclasping moves. This implies the existence of generating families f_- for Λ_- and f_+ for Λ_+ and an immersed GF-compatible cobordism from (Λ_-, f_-) to (Λ_+, f_+) with two double points.

The remainder of this section is devoted to proving Theorems 4 and 5. We begin by recalling the following definition:

Definition 18. Let (Λ, f) be a Legendrian knot with polynomial

$$\Gamma_f(t) = c_n t^{-n} + ... + c_1 t^{-1} + c_0 t^{-0} + t + c_0 t^0 + c_1 t^1 + ... + c_n t^n. \quad (24)$$

A **minimal immersed GF-compatible disk filling** is one with genus 0 and c_k immersion points of index k for all $k \in \{0, ..., n\}$.

For Legendrian knots, the following theorem was proven by Melvin-Shrestha [25] in terms of Legendrian Contact Homology, and by Bourgeois-Sabloff-Traynor [4] for generating families. The latter also gives a higher dimensional analog of this theorem.

Theorem 16 (Theorem 1.2 in [25], Theorem 1.2 in [4]). *For any polynomial Γ satisfying one-dimensional duality, there exists (Λ, f) such that $\Gamma_f = \Gamma$.*

The proof of this theorem is entirely constructive and is based on the building block T_d in Figure 34. It can be verified that this twist Legendrian knot has a generating family with polynomial $\Gamma(t) = t^{-d} + t + t^d$. Using the fact that for two Legendrians (Λ, f), (Λ', f'),

$$\Gamma_{f\#f'}(t) = \Gamma_f(t) + \Gamma_{f'}(t) - t, \tag{25}$$

we can take connect sums of Legendrian knots of the type T_d to achieve any desired polynomial.

We are now ready to prove Theorem 4, which is a simple extension of the argument above. Recall that Theorem 4 states that given a polynomial, there exists a Legendrian having that polynomial which has a minimal immersed GF-compatible disk filling.

Proof of Theorem 4. Construct a Legendrian (Λ, f) with polynomial $\Gamma(t)$ in the method described above. A normal ruling can be constructed with no switches at the clasp and a switch on each of the twist crossings. Perform an unclasping move at each clasp and a series of Reidemeister 1 moves for all of the twists to obtain the $(-1, 0)$-unknot, which is known to have an embedded disk GF-compatible filling. □

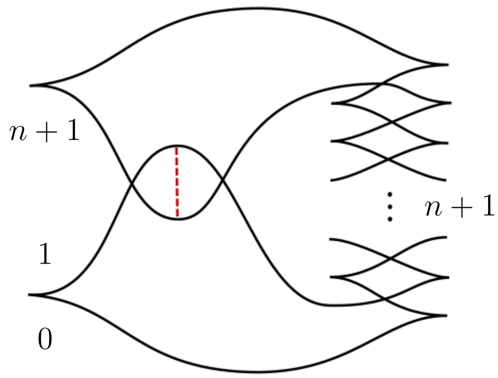


Figure 34: The knot T_d has polynomial $t^{-d}+t+t^d$.

Theorem 5 gives us ways of constructing new fillings from existing ones. The first part allows us to create a new filling with the same genus and an additional pair of immersion points. The second allows us to create a new filling with higher genus at the expense of an additional immersion point. Together, these prove the existence of the entire upper diagonal of the lattice in Figure 7. The proof of this theorem is an application the following fact from [4]: performing a 0-surgery on an embedded 2-dimensional Legendrian gives rise to an embedded 2-dimensional Legendrian. This surgery is performed by attaching a 1-handle, shown in Figure 35, in a 1-attaching region (i.e. a cusp edge, shown in green) on the Legendrian along a core disk (shown in red). After lifting the given Lagrangian filling $(\mathcal{L}, F)$ to a 2-dimensional Legendrian, we can perform this move locally so that the boundary is not affected. Projecting back down to the Lagrangian gives us a new filling that is still GF-compatible with the Legendrian (Λ, f).

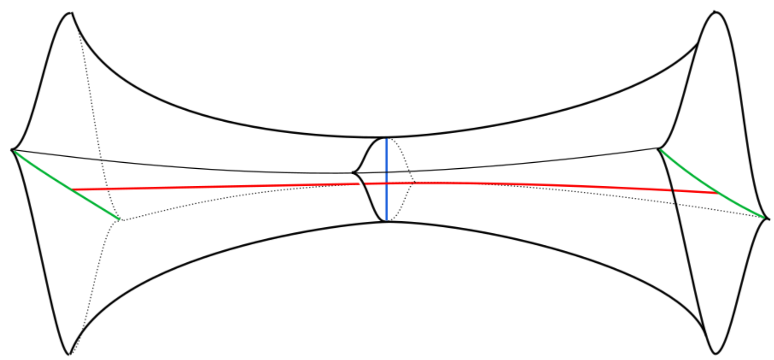


Figure 35: A 1-handle with one Reeb chord, shown in blue. The 1-attaching region is shown in green and the core disk is shown in red.

Proof of Theorem 5. Consider the Legendrian lift $\Lambda_{\mathcal{L}}$ of $(\mathcal{L}, F)$ to $\mathbb{R} \times T^*(\mathbb{R}_+ \times M)$. Since $\mathcal{L}$ is cylindrical over Λ, there exists a neighborhood in $\mathcal{L}$ of the rightmost cusp c of Λ such that in the Legendrian lift $\Lambda_{\mathcal{L}}$, this neighborhood consists of a cusp edge as depicted in Figure 36. Choose a core disk between a point in this neighborhood and a point on a cusp edge in a "Legendrian flying saucer" as in Figure 37. After performing the 0-surgery, the new Legendrian has two new Reeb chords: one on the 1-handle and one on the flying saucer. Now, project this 2-dimensional Legendrian in $\mathbb{R} \times T^*(\mathbb{R}_+ \times M)$ to a 2-dimensional Lagrangian $(\mathcal{L}', F')$ in $T^*(\mathbb{R}_+ \times M)$. The two Reeb chords result in two immersed double points in $(\mathcal{L}', F')$. Since the modifications made on $\Lambda_{\mathcal{L}}$ were done locally and away from the boundary, the filling $(\mathcal{L}', F')$ is still GF-compatible with (Λ, f).

The proof the second part of this theorem follows a similar structure. Instead of attaching the 1-handle along a core disk connecting to a flying saucer, create a core disk between two points in the lifted neighborhood of c. The resulting 2-

dimensional Legendrian will have additional genus and one additional Reeb chord. Consequently, the Lagrangian projection $(\mathcal{L}', F')$ will have additional genus and one additional immersed double point. $\square$

Remark 19. Alternatively, we could use the combinatorial moves described in the previous section on any right cusp of the front diagram of Λ, as in Figures 38 and 39. However, these moves do not guarantee that $(\mathcal{L}', F')$ is GF-compatible with (Λ, f). It would however, show that there exists a generating family f' for Λ and a GF-compatible immersed filling $(\mathcal{L}', F')$ of (Λ, f') such that either (a) $(\mathcal{L}', F')$ has the same genus and two additional immersion points, one of index k and one of index $k+1$ or (b) with genus $g+1$ and one additional immersion point of index 1.

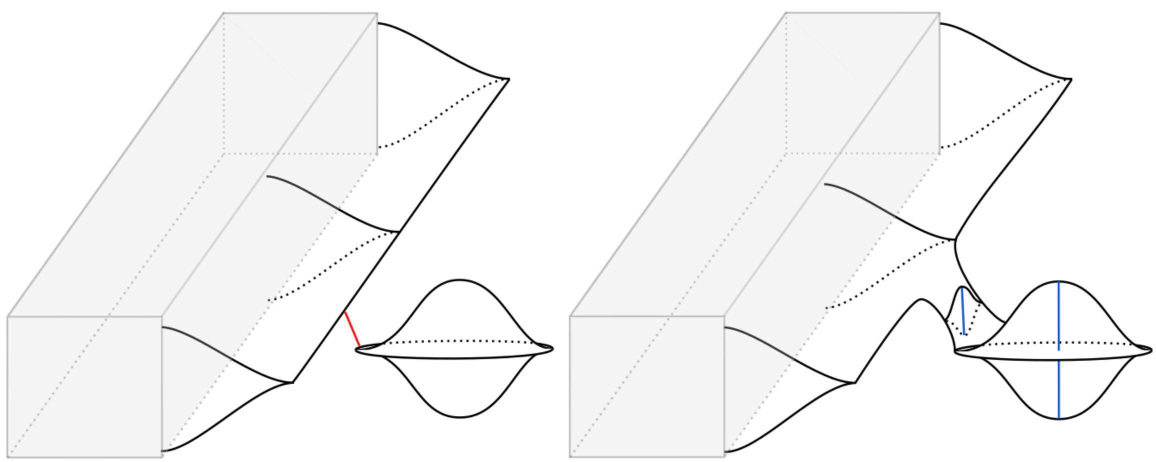


Figure 36: Performing a 0-surgery on the cusp edge of $\Lambda_{\mathcal{L}}$ with a Legendrian flying saucer as in the proof of Theorem 5.

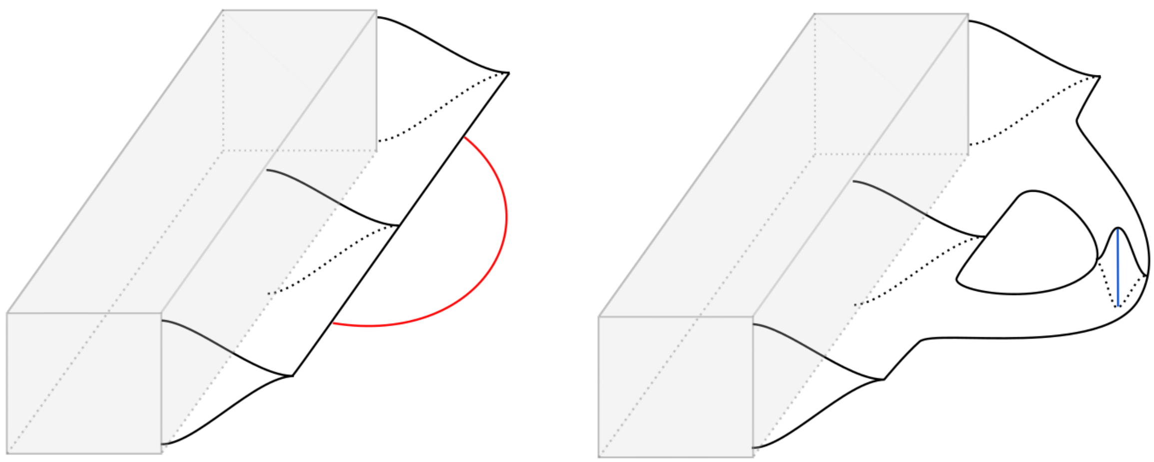


Figure 37: Performing a 0-surgery along the attaching curve extending between two points of the cusp edge of $\Lambda_{\mathcal{L}}$ as in the proof of Theorem 5.

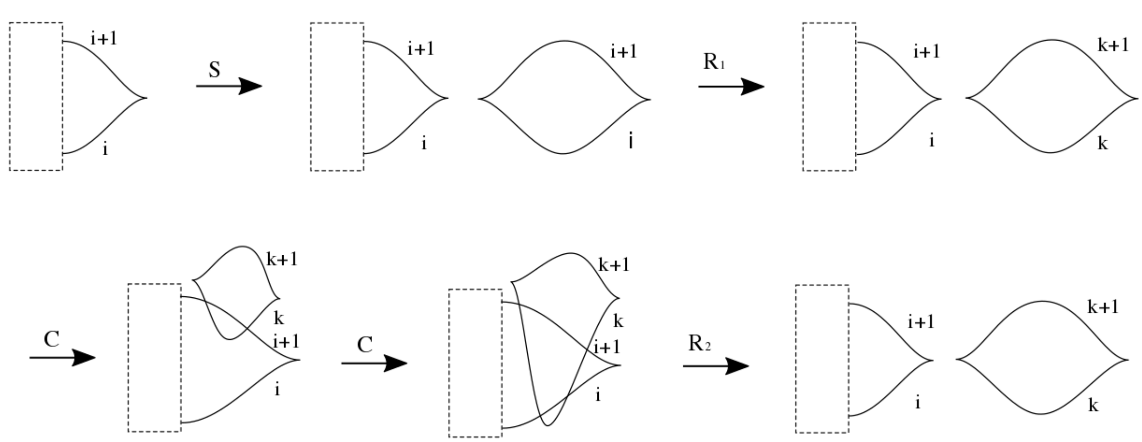


Figure 38: Alternate construction for Theorem 5 to create a new GF-compatible filling with two additional immersed double points.

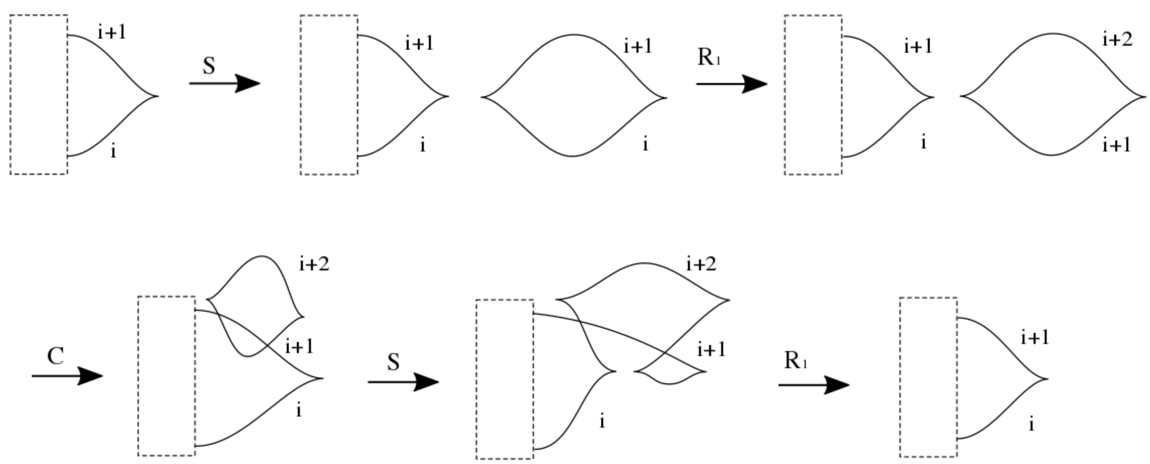


Figure 39: Alternate construction for Theorem 5 to create a new GF-compatible filling with higher genus and one additional immersed double point.

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March 30, 2018

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Education

- **Bryn Mawr College** — Bryn Mawr, PA
  Ph.D Mathematics (in progress) — *Expected May 2018*
  M.A. Mathematics — *May 2015*
- **Ramapo College of New Jesery** — Mahwah, NJ
  B.S. Mathematics — *May 2012*
  - Graduated Summa Cum Laude with honors and distinction, with a 3.99 GPA
  - Studied abroad at Kingston University in London, UK, July 2009
  - Honor Societies: Alpha Lambda Delta, Pi Mu Epsilon, Golden Key Honor Society

Appointments

- **Penn State Brandywine** — Media, PA
  Assistant Teaching Professor — *Beginning August 2018*

Research Experience

- *Immersed Lagrangian Fillings of Legendrian Submanifolds via Generating Families* — *November 2015 - Present*
  Ph.D. Dissertation
  Advisor: Lisa Traynor
- *Randomly Generated Legendrian Knots* — *September 2016- Present*
  Joint with: Lisa Traynor, Samantha Kacir (undergraduate)
- *Geospatial Pattern Detection, Recognition, and Prediction of Natural Systems* — *August 2017- Present*
  Joint with: Bobak Karimi, Hassan A. Karimi
- *Minimal Genus Lagrangian Caps of Legendrian Knots* — *May 2014 - May 2015*
  Master's Thesis
  Advisor: Lisa Traynor
- *Minimal Degree Parameterizations of the Trefoil and Figure Eight Knots* — *May 2011 - August 2012*
  Undergraduate Research
  Advisor: Donovan McFeron

Teaching Experience

- **Adjunct Professor** — Ocean County College
  Calculus II — *Summer 2016*
  - Taught a summer course which involved creating my own syllabus, lectures, written and online assignments, exams, and projects.

- **Teaching Assistant** — Bryn Mawr College
  Undergraduate Courses — *2012-Present*
  - Taught a section of Calculus II using my own syllabus, assignments, exams and projects.
  - TA Courses: Multivariable Calculus, Linear Algebra, Knot Theory, Probability, Analysis, Abstract Algebra, Combinatorics, Topology
    * Grade homework assignments and exams.
    * Lead problem sessions each week and give occasional lectures.
- **Mathematics Tutor** — Ramapo College
  Undergraduate Courses — *2009-2012*
  - Courses: Elementary Algebra, Transitional Mathematics, Math for the Modern World, Calculus
- **Reading and Writing Workshop Instructor** — Ramapo College
  High School English — *Fall 2011*
  - Planned and executed weekly lesson plans for this weekly after-school program at Ramsey High School, in collaboration with Ramapo College.

Awards, Grants & Honors

- ***McPherson Award for Excellence and Service to the Community*** — *May 2017*
- ***Margaret LaFoy Rossiter and Mabel Gibson LaFoy Fellowship*** — *2016/17 Academic Year*
- ***Bryn Mawr College Teaching Assistantship and Tuition Award*** — *September 2012-Present*
- ***Dean's Certificate in Pedagogy*** — *May 2016*
- ***Art Exhibitor at the Joint Mathematics Meeting***
  - Painting entitled *Square Root of Two* — *January 2015*
  - Sculpture entitled *Legendrian Unknot* — *January 2014*
- ***Dean's Award for Outstanding Achievement in Mathematics*** — *April 25, 2012*
- ***Ramapo College Presidential Scholarship*** — *September 2008-May 2012*

Presentations

- ***Joint Mathematics Meetings***
  - MAA General Contributed Paper Session on Topology — *January 12, 2017*
  - Women in Symplectic and Contact Geometry and Topology, II
  - AWM Workshop: Poster Presentations by Women Graduate Students
- ***Immersed Lagrangian Fillings of Legendrian Submanifolds***
  - Geometry/Topology Seminar, Massachusetts Institute of Technology — *November 20, 2017*
- ***Random Legendrian Knots***
  - EPaDel MAA Meeting, Shippensburg University — *November 18, 2017*
- ***Introduction to Knot Theory***
  - Mathematics Colloquium, Albright College — *November 13, 2017*

- ***Immersed Lagrangian Fillings of Legendrians via Generating Families***
  - ANR Cospin, École Normale Supérieure de Lyon *October 30, 2017*
- ***Fillings of Legendrian Knots: Obstructions and Constructions***
  - Tetrahedral Geometry and Topology Seminar *October 13, 2017*
- ***Immersed Lagrangian Fillings of Legendrian Knots***
  - Graduate Student Geometry and Topology Conference, Michigan State *April 8, 2017*
- ***Knot Polynomials and the Information they Encode***
  - EPaDel/NJ MAA Meeting, Kutztown University *April 1, 2017*
- ***Legendrian Knots and their Fillings***
  - Graduate Student Research Symposium, Bryn Mawr College *March 29, 2017*
- ***Immersed Lagrangian Fillings of Legendrian Knots***
  - Tech Topology Conference, Georgia Tech *December 10, 2016*
- ***Lagrangian Fillings of Legendrian Knots***
  - EPaDel/NJ MAA Meeting, Villanova University *November 12, 2016*
- ***The Wizardry of Whitney's Theorem II***
  - Part 1: PACT Seminar, Bryn Mawr College *January 26, 2016*
  - Part 2: PACT Seminar, Bryn Mawr College *February 2, 2016*
  - Distressing Math Collective, Bryn Mawr College *January 28, 2016*
- ***Minimal Genus Lagrangian Caps of Legendrian Knots***
  - Master's Thesis Defense, Bryn Mawr College *May 7, 2015*
  - GSAS Poster Session, Bryn Mawr College *April 7, 2015*
- ***Honeycombs and Infinite Soccer Balls: Introduction to Non-Euclidean Geometries***
  Workshop geared towards grade school and beginning college students.
  - Math Teachers Circle, The Philadelphia School, Philadelphia, PA *March 24, 2015*
  - Guest Lecture, Westwood Regional High School, Westwood, NJ *June 9, 2014*
  - Temple Mathematics Circle, Temple University *March 29, 2014*
  - Guest Lecture, Ocean County College, Toms River, NJ *March 14, 2014*
  - Guest Lecture, Ramapo College, Mahwah, NJ *March 11, 2014*
  - CATALYST, Swarthmore College *March 24, 2013*
- ***The Wizardry of Whitney's Theorem***
  - Distressing Math Collective, Bryn Mawr College *October 9, 2014*
  - EPaDel MAA Meeting, University of Scranton *April 26, 2014*
- ***The Braid Group and Other Tangled Topics***
  - Distressing Math Collective, Bryn Mawr College *January 30, 2014*
- ***What's Behind Door Number One?: How not to get zonked in 'Let's Make a Deal' and 'Deal or No Deal'***
  - Distressing Math Collective, Bryn Mawr College *September 19, 2013*
- ***Math and Crafts: Fun with Non-Euclidean Geometry***

– Distressing Math Collective, Bryn Mawr College *April 4, 2013*

- ***Minimal Degree Parameterizations of the Trefoil and Figure Eight Knots***
  - Distressing Math Collective, Bryn Mawr College *November 1, 2012*
  - Theoretical and Applied Sciences Research Symposium, Ramapo College *April 18, 2012*
  - Spuyten Duyvil Undergraduate Mathematics Conference, Ramapo College *April 14, 2012*
  - Honors Symposium, Ramapo College *April 4, 2012*
  - Joint Mathematics Meeting, Boston, Massachusetts *January 5, 2012*
- ***Models of the Hyperbolic Plane***
  - Math Club Meeting for Undergraduate Speakers, Ramapo College *April 19, 2011*

Service

- **Graduate Student Association** Bryn Mawr College
  Co-Chair *September 2015–August 2017*
  Mathematics Department Representative *September 2014–Present*
- **AMS Graduate Student Chapter** Bryn Mawr College
  President *September 2017–Present*
- **GGSM Professional Development Seminars** Bryn Mawr College
  Organizer *September 2016–Present*
- **Girls Go STEM** Girl Scouts of America
  Exhibitor *March 17, 2018*
  - Represented the Association for Women in Mathematics at this event by presenting various engaging math activites to young girls.
- **Math Appreciation Week** Bryn Mawr College
  Organizer *April 2016/2017*
  - Performed a Mathematical Magic Show for undergraduate students.
  - Led a Mathematical Knitting Circle for the Distressing Math Collective.
  - Prepared and hosted a "Math Pictionary" game for undergraduate students.
- **Math Counts Competition** Ocean County College
  Volunteer Grader *January 30, 2016*
- **Outreach to High Schools**
  Biotechnology High School *September 24-25, 2015*
  - Led discussions through Skype about whether math was invented or discovered as part of the school's "Theory of Knowledge" class.

  Westwood Regional High School *June 9, 2014*
  - Led a hands-on workshop about Non-Euclidean geometries.
- **Math Teachers Circle** The Philadelphia School
  Workshop Leader *March 24, 2015*
  - Planned and led a workshop to teach math teachers about Non-Euclidean geometry.
- **Temple Mathematics Circle** Temple University
  Workshop Leader

  - Planned and led workshops for advanced 5th-8th graders:
    * "The Mathematics of Coloring" *March 29, 2014*
    * "Euclidean and Non-Euclidean Tessellations" *February 21, 2015*

- **Graduate Fair Representative** Joint Mathematics Meetings
  San Antonio, TX *January 2015*
  Baltimore, MD *January 2014*

- **CATALYST Conference** Swarthmore College
  Workshop Leader *March 24, 2013*
  - Program to encourage female middle school students to pursue STEM fields.
  - Planned and led a workshop entitled "Honeycombs and Infinite Soccer Balls: Euclidean and Non-Euclidean Tessellations".

- **Spuyten Duyvil Undergraduate Mathematics Conference** Ramapo College
  Conference volunteer and presenter *April 14, 2012*

- **Pi Mu Epsilon Mathematics Honor Society** Ramapo College
  Vice-President and Secretary *March 2011–March 2012*

Funded Workshops and Conferences

- **Joint Mathematics Meeting** Various Locations
  San Diego, CA *January 10-13, 2017*
  San Antonio, TX *January 10-13, 2015*
  Baltimore, MD *January 15-18, 2014*
  Boston, MA *January 4-7, 2012*

- **Symplectic Geometry in Lyon (Sikorav 60)** Lyon, FR
  Université Lyon 1 and École Normale Supérieure *November 2-4, 2017*

- **ANR Cospin** Lyon, FR
  École Normale Supérieure *October 30-1, 2017*

- **Graduate Student Geometry and Topology Conference** East Lansing, MI
  Michigan State University *April 8-9, 2017*

- **Tech Topology Conference** Atlanta, GA
  Georgia Tech *December 9-11, 2016*

- **Topology Students Workshop** Atlanta, GA
  Georgia Tech *June 6-10, 2016*

- **Dusa McDuff's 70th Birthday Conference** New York City, NY
  Columbia University *March 17-20, 2016*

- **AWM Research Symposium** College Park, MD
  University of Maryland *April 11-12, 2015*

- **Redbud Topology Conference** Fayetteville, AR
  University of Arkansas *March 8-10, 2013*

- **Women and Mathematics Program** Princeton, NJ
  Institute for Advanced Study *May 14-25, 2012*

- **Garden State Undergraduate Mathematics Conference** Newark, NJ
  Essex County College *April 2, 2011*

Regular Research Seminars

- **Philadelphia Area Contact Topology (PACT)** — Bryn Mawr, PA
  Bryn Mawr College — *Weekly*
- **Philadelphia Area Topology (Contact/Hyperbolic) Seminar (PATCH)** — Philadelphia, PA
  UPenn, Temple University, Haverford College and Bryn Mawr College — *Monthly*
- **EPaDel MAA Meetings** — Various Locations
  Temple University — *March 24, 2018*
  Shippensburg University — *November 18, 2017*
  Kutztown University — *April 1, 2017*
  Villanova University — *November 12, 2016*
  Scranton University — *April 26, 2014*
  St. Joseph's University — *November 9, 2013*