Relative Khovanov-Jacobsson Classes for Spanning Surfaces

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Relative Khovanov-Jacobsson Classes

for Spanning Surfaces.

by

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Abstract

We define the Khovanov-Jacobsson class for a properly embedded surface in the 4-ball, an element of the Khovanov homology of its boundary link in the 3-sphere. We then develop general non-triviality criteria for Khovanov homology classes, and use these to distinguish the Khovanov-Jacobsson classes of various families of surfaces. Among these are pairs of distinct slice disks for pretzel knots, and the first known examples of pairs of Seifert surfaces of equal genus for links in the 3-sphere that remain distinct when pushed into the 4-ball.
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1. Introduction

Khovanov homology assigns to each link $L$ in $S^3$ a bi-graded finitely generated $R$-module $Kh(L)$, where $R$ is a ring with unit, typically equal to $\mathbb{Z}$, $\mathbb{Z}_2$, or $\mathbb{Q}$. It is known that $Kh$ is functorial, in the sense that a link cobordism induces a homomorphism (i.e. an $R$-linear map) between the homologies of its boundary links. More precisely, each smooth, compact, oriented surface $S$ in $B^4$ whose interior lies between two concentric $S^3$’s gives rise to a $R$-linear map from the homology of the inner boundary link to the homology of the outer one. Magnus Jacobsson [5] proved that this map is invariant up to sign under isotopies of $S$ that keep the boundary fixed.

In particular, if $S$ is a properly embedded surface in $B^4$ bounded by a link $L$ in $S^3$, then $S$ can be viewed as a cobordism from the empty link $\emptyset$ to $L$. Such a surface will be called a spanning surface for $L$. For example, any Seifert surface for $L$ in $S^3$ becomes a spanning surface for $L$ after pushing its interior into $B^4$. Given that $Kh(\emptyset) = R$, we define the Khovanov-Jacobsson ($KJ$) class of $S$ to be the homology class that is the image in $Kh(L)$ of the unit $1 \in R = Kh(\emptyset)$ under the map induced by $S$.

For closed surfaces, i.e. when $L$ is empty, this invariant was studied by Jacobsson, Rasmussen [8], Tanaka [9], and others, and was shown
to be uninteresting: it is always equal to 2 for tori, and 0 otherwise. For bounded surfaces, it is (until now) largely unexplored.

It was previously known that a link could admit multiple spanning surfaces that are homeomorphic but not isotopic in $S^3$. It was not known, however, if these surfaces embedded in $S^3$ could remain non-isotopic after being pushed into $B^4$. We provide an example of a family of links that each admit two spanning surfaces in $S^3$ of equal genus that remain distinct even in the 4-ball, as detected by $KJ$.

In addition to this example, we provide a complete classification of $KJ$ invariants for surfaces spanning unlinks. This leads to several statements regarding sliceness and $KJ$ invariants of surfaces spanning slice knots. We prove that $KJ$ cannot detect local knottedness but show that KJ can sometimes distinguish between two distinct slices for a given slice knot.

Each of these results requires proving that particular chains in the Khovanov complex represent non-trivial homology classes. To accomplish this, we provide a set of non-triviality criteria for simple classes in Khovanov homology. Andrew Elliott also produced several of these results in [3], in which he applied them to his study of homological width. Although our statements were developed independently of Elliot’s, we credit [3] for insight in fixing an initial flawed proof of Theorem 8. We
also provide a general framework for extending these results beyond simple states, via representation maps.

The basic definitions presented in the first half of this paper roughly follow the approach set out by Bar-Natan in [1]. We alter Bar-Natan’s definitions, however, to avoid the multiple layers of category theory that he uses to justify his invariance statement. Instead we pose the cube of resolutions $C(D)$ as an honest (albeit infinitely generated) chain complex of $R$-modules. In our context the invariance proofs remain identical to Bar-Natan’s, except that we need Theorem 2 in order to support the transition from Reidemeister tangle cube maps to maps of link diagram cubes.

1.1. **Layout of Paper.** In Section 2 we will define the cube of resolutions $C(D)$ for a diagram and set the stage for the proof of its invariance, which will follow in Section 3. In Section 4 we will relate this definition to the original theory $Kh$ defined by Khovanov. We will present an example calculation of $Kh$ in Section 5. Those readers familiar with Khovanov’s theory may choose to turn directly to Section 6, in which we will present conditions and techniques for determining non-triviality of specific classes in $Kh(L)$. We will define the relative Khovanov-Jacobsson invariant $KJ$ in Section 7 and analyze $KJ(S)$ for several classes of surfaces in Sections 7.3 and 8.
2. The Cube of Resolutions for a Link Diagram

Let $L$ be an oriented link in $S^3$, with a diagram $D$ of $n$ crossings. For a given ordering of the $n$ crossings, associate to $D$ the Cube of Resolutions for $D$, denoted $C(D)$, which is the topological structure constructed as follows:

Consider the standard $n$-dimensional cube with vertices $v \in \{0, 1\}^n$. Assign to each vertex $v = (v_1, v_2, \ldots, v_n)$ the planar 1-manifold $M_v$ that results from resolving each crossing in $D$ such that the $i^{th}$ crossing is resolved as a $v_i$-resolution, according to the following rule:

```
  \begin{array}{cccc}
    ( & ) & ( & ) \\
    ( & ) & ( & ) \\
  \end{array}
```

Such a 1-manifold will be called a smoothing of $D$. Note that the smoothing at $v_i$ is independent of the orientation on $L$.

Impose on vertices a notion of height, defined by

$$h(v) = \left( \sum_{i=1}^{n} v_i \right) - n_-,$$

where $n_-$ is the number of negative crossings in $D$. Extend this height to smoothings by $h(M_v) = h(v)$. Note that the height of a smoothing $M_v$ can be calculated directly from $M_v$ by counting the number of

```
  \begin{array}{cccc}
    ( & ) & ( & ) \\
    ( & ) & ( & ) \\
  \end{array}
```
crossings in $M_v$ that are resolved according to the 1 rule and then subtracting the number of negative crossings in $D$. Since $n_-$ is dependent on the relative orientations on the components in $L$, $h$ is orientation sensitive if $L$ has multiple components.

Connect each pair of smoothings in adjacent heights by weighted cobordisms between them: Each pair of smoothings at vertices $u, v$ in \{0, 1\}^n that differ in exactly one coordinate is connected by a directed, orientable cobordism $E_{v,u}$ in $\mathbb{R}^2 \times I$ for which $\partial(E_{v,u}) \cap (\mathbb{R}^2 \times \{0\}) = M_u$ and $\partial(E_{v,u}) \cap (\mathbb{R}^2 \times \{1\}) = M_v$.

The single coordinate that changes between $u$ and $v$ indicates that one resolution has been changed from 0 to 1 between $M_u$ and $M_v$. This resolution change corresponds to either a merger of two circles into one or a split of one circle into two within a neighborhood $B^2$ of the crossing. The inside of $B^2 \times I$ in $E_{v,u}$ is therefore the appropriate simple saddle that reflects this merger/split. The portion of $E_{v,u}$ outside of the cylinder $B^2 \times I$ is an identity cobordism. This $E_{v,u}$ is directed by increasing height and weighted with a coefficient $(-1)^k$, where $k$ is the number of 1's in $u$ to the left of the single coordinate that changes between $u$ and $v$. 

\[ E_{v,u} \]

\[ M_u \]

\[ M_v \]
For vertices $u$ and $v$ that differ in more than one coordinate, define $E_{v,u} = 0$, the cobordism with coefficient 0.

\[ h = 1 - 2 = -1 \]

**Figure 1.** $C(D)$ for the left-handed Hopf link.

Given any oriented knot or link diagram $D$ we can construct such a cube $C(D)$. This topological structure contains just as much information as the original link diagram; given $C(D)$, we can re-create the diagram $D$, up to relative orientation. As such, $C(D)$ is no closer to providing general information about the link $L$ than is $D$. In an effort to distill from $C(D)$ some useful topological information, we will first modify our definition slightly, viewing $C(D)$ as a chain complex of $R$-modules. The definitions and invariance proof will be presented using a general ring $R$, but in Section 4 we will want to assume that
2 is invertible. From Section 6 on we will use $R = \mathbb{Q}$ coefficients in all calculations and examples.

For a given smoothing $M$, let $V_M$ be the $R$-module generated by the set of all cobordisms $S$, smoothly embedded in $\mathbb{R}^2 \times I$, for which $\partial S = M \subseteq \mathbb{R}^2 \times \{1\}$, taken up to boundary preserving isotopy. A general element in $V_M$ is therefore an $R$-linear combination of such nullbordisms in $\mathbb{R}^2 \times I$, with matching boundaries. In a slight abuse of notation, we will refer to such an $R$-linear combination of cobordisms simply as a cobordism.

Define the height of a cobordism $S$ in $V_M$ by $h(S) = h(M)$.

Suppose $E$ is a cobordism in $\mathbb{R}^2 \times I$ for which $E \cap (\mathbb{R}^2 \times \{0\}) = M$ and $E \cap (\mathbb{R}^2 \times \{1\}) = N$. Then $E$ can be viewed as a module homomorphism $E : V_M \to V_N$, taking a cobordism $S \in V_M$ to the composite cobordism $E \circ S \in V_N$. The coefficients on $E$ and $S$ multiply. Again, all cobordisms are taken up to boundary preserving isotopy. Extend this definition linearly to allow for $R$-linear combinations of cobordisms that have identical boundaries.

If we consider the picture of each manifold $M$ in $C(D)$ to be shorthand for $V_M$ and view the edges $E_{v,u}$ as module homomorphisms, as described in the paragraph above, then the cube $C(D)$ becomes an honest chain complex. The height levels are given by
\[ C^j(D) = \bigoplus_{h(M)=j} V_M \]

and the boundary maps can be organized as matrices

\[
d^j = \begin{pmatrix}
E_{v_1,u_1} & E_{v_1,u_2} & \cdots & E_{v_1,u_m} \\
E_{v_2,u_1} & E_{v_2,u_2} & \cdots & E_{v_2,u_m} \\
\vdots & \vdots & \ddots & \vdots \\
E_{v_n,u_1} & E_{v_n,u_2} & \cdots & E_{v_n,u_m}
\end{pmatrix},
\]

where the \( u_i \) are those vertices at height \( j \) and the \( v_i \) are those at height \( j+1 \). Note that each non-zero entry in the \( i^{th} \) column corresponds to a unique 0-resolved crossing in \( M_{u_i} \). In this context, the cobordism \( E_{v,u} \) is said to be the edge differential connecting \( M_u \) and \( M_v \).

This matrix structure formalizes the desired composition properties for the cobordisms in \( C(D) \). In practice, however, the notation becomes rather cumbersome. We will often depict \( C(D) \) in the picture form presented in Figure 1 if doing so will not lead to confusion.

In order to see that \( d^{i+1} \circ d^i \) is the zero matrix, we need only observe that each face in \( C(D) \) forms an anti-commutative square. The composite cobordism found around one side of a given face will contain exactly two saddles. If we isotope this composite surface to reverse the order of the two saddles, we end up with the composite cobordism.
found around the other side of that face. Our choice of coefficients guarantees that these two isotopic cobordisms will always have opposite signs, meaning that they will cancel when computing $(d)^2$.

As observed above, $C(D)$ still retains too much information to itself be an invariant of $L$. In order to extract a meaningful invariant, we follow Bar-Natan’s approach in [1], imposing a set of local relations on the cobordisms in $C(D)$. We will present the standard relations here and then explore the implications of these choices in Section 4.

\[ (S) \quad \begin{array}{c} \circlearrowright \circlearrowleft \\ \hookrightarrow \times 0 \end{array} \quad (T) \quad \begin{array}{c} \circlearrowleft \circlearrowright \\ \hookrightarrow \times 2 \end{array} \]

**Figure 2.** Sphere and Torus relations.

The first two relations that we impose on cobordisms in the cube are the Sphere and Torus relations shown in Figure 2. The first states that if a cobordism $E$ contains a free-floating sphere, then $E$ is equivalent to the zero cobordism, meaning the zero linear combination cobordism. The second states that a cobordism $E$ that contains a free-floating torus is equivalent to $2E'$, where $E'$ is $E$ with the torus removed.

The third relation that we impose is the Four Tube relation, which says that if a sphere $S$ intersects the surface $E$ in exactly four circles, as shown in Figure 3, then the given equation holds. Each surface in
this equation represents a surface formed from $E$ with the interior of $S$ replaced by the surface shown.

Figure 3. The Four Tube relation ($4t$).

From this point forward we will take $C(D)$ to be the chain complex given by $(C^j, d^j)$, as described above, in which the cobordisms are all taken up to boundary preserving isotopy and equivalence under the set of local relations $l = \{(S), (T), (4t)\}$.

2.1. Cubes of Tangles. As is often the case, before making any claim of invariance we must first develop a relative version of the theory. Therefore we need a notion of the cube of resolutions $C(T)$ for a tangle diagram $T$ contained within a ball $B$.

When building $C(T)$, everything remains exactly the same as in the link case, except that the smoothings and morphisms all have boundaries. In particular, each smoothing $M$ has $\partial M = \partial T$ and each edge cobordism has $\partial E_{v,u} = M_u \cup M_v \cup (\partial M_u \times I)$. The $R$-modules $V_M$ are generated by surfaces $S \subset B \times I$ for which $\partial S \cap (B \times \{1\}) = M$ and $\partial S \subset (B \times \{1\}) \cup (B \times (0,1))$. All cobordisms in $C(T)$ are taken up
to the local relations in $l$ and up to isotopies that preserve their 0 and 1 frames and isotope the vertical portions of their boundaries within $\partial B \times (0,1)$.

2.2. Chain Maps and Homotopy Equivalence. We now need to establish our notion of equivalence between cubes. Begin by defining chain maps in our setting.

Given complexes $C_1$ and $C_2$, a chain map $F : C_1 \rightarrow C_2$ is a collection of maps $F^i : C^i_1 \rightarrow C^{i+k}_2$ for which $F^{i+1}d^i_1 = d^{i+k}_2F^i$ for each $i$. If $k = 0$ we say the chain map $F$ is height preserving. Otherwise, we describe $F$ as being a chain map of degree $k$.

**Example 1.** Suppose $D$ and $D'$ are oriented link diagrams that are identical (up to orientation) outside of a ball $B$, inside of which they contain the opposite crossingless two-strand tangles. Take the crossings in $D$ and $D'$ to have the same order.

Each pair of corresponding smoothings $M_u$ in $C^i(D)$ and $M'_u$ in $C^i(D')$ can be connected by a cobordism $E_u$ defined to be the identity cobordism except inside $B \times I$, which contains a simple saddle. The collection of diagonal matrices $F^i$ whose diagonal entries are given by the $E_u$ give an example of a chain map $F : C(D) \rightarrow C(D')$. This $F$ is referred to as the chain map induced by a saddle in $B$. 
The fact that $F$ is a chain map follows directly from the fact that for each pair of adjacent vertices $u, v \in C(D)$, we have $E'_{v,u}E_u = E_vE_{v,u}$. This is to say that the saddle portion of each $E_u$ occurs inside $B$ while the saddle portion of $E_{v,u}$ occurs outside of $B$, meaning that the two saddles can be time re-ordered. The coefficients of $E_{v,u}$ and $E'_{v,u}$ in $d$ and $d'$ will always match since the crossings in $C(D)$ and $C(D')$ are assumed to have the same order.

A word of caution: This example has been described without any reference to orientations. The orientations on $D$ and $D'$ might agree everywhere, in which case the resulting $F$ will be a height preserving chain map. If the orientations do not agree everywhere outside of $B$, however, the chain map $F : C(D) \to C(D')$ will involve a shift in height by some $k$, which is dictated by the choice of orientation on $D'$. We will revisit this example again in Section 6.4.

Given this definition of chain maps of cubes, we are ready to define chain homotopy equivalence. Recall the following standard definition, expressed here in our current notation. Two cubes $C(D_1)$ and $C(D_2)$ are chain homotopy equivalent if there exist height preserving chain maps $F$ and $G$ between them such that the compositions $F \circ G$ and $G \circ F$ are each homotopic to the appropriate identities. This is to say
that there exist maps $h_i : C(D_i) \to C(D_i)$ of degree $-1$ for which

$$G^i \circ F^i - I = d_1^{i-1} h_1^i + h_1^{i+1} d_1^i$$

$$F^i \circ G^i - I = d_2^{i-1} h_2^i + h_2^{i+1} d_2^i.$$

All maps are shown in Figure 4.

\[\cdots \quad \begin{array}{ccccccc}
C_{1-1}^i & \overset{d_1^{i-1}}{\longrightarrow} & C_1^i & \overset{d_1^i}{\leftarrow} & C_1^{i+1} & \overset{G_{i+1}}{\longrightarrow} & \cdots \\
\downarrow G_{i-1} & & \downarrow F_i & & \downarrow G_i & & \downarrow F_{i+1} \\
C_{2-1}^i & \overset{d_2^{i-1}}{\longrightarrow} & C_2^i & \overset{d_2^i}{\leftarrow} & C_2^{i+1} & \overset{G_{i+1}}{\longrightarrow} & \cdots
\end{array}\]

**Figure 4.** Chain homotopy equivalence.

This provides the notion of equivalence of cubes that we will need in order to formulate our invariance statement.

2.3. **Lifting Tangle Maps.** We can now use $C(T)$ to compare two almost identical link diagrams.

**Theorem 2.** Suppose $D_1$ and $D_2$ are two link diagrams that agree outside of some disk $B$, inside of which they contain tangle diagrams $T_1$ and $T_2$, respectively. Let $F : C(T_1) \to C(T_2)$ be a chain map. Then

1. $F$ can be lifted to a chain map $\hat{F} : C(D_1) \to C(D_2)$. 
Furthermore, if $F$ (and $G$) define a chain homotopy equivalence between $C(T_1)$ and $C(T_2)$, then the lifts $\widehat{F}$ and $\widehat{G}$ yield a chain homotopy equivalence between $C(D_1)$ and $C(D_2)$.

This result is useful for two reasons. The first part allows us to define a chain map on $C(D)$ by describing only the “interesting part” of the map in terms of a map on $C(T)$ for some tangle $T$ in $D$. From this point forward, any map on $C(D)$ that is described by its behavior on a tangle $T$ within $D$ will be assumed to be extended, by way of Theorem 2(1), to a map on the rest of $C(D)$. We will abuse this ability extensively in future sections.

The second part of Theorem 2 allows us to prove our invariance statement in Section 3 by looking only at the tangle maps induced by the Reidemeister moves, with the knowledge that the induced chain homotopy equivalences can always be extended outside of their respective tangles, as needed.

Proof: Let $F, G, h_1,$ and $h_2$ be the maps involved in the equivalence between $C(T_1)$ and $C(T_2)$. Let the ordering of crossings in $T_1$ and $T_2$ be given, with corresponding differentials in $C(T_1)$ and $C(T_2)$ denoted $d_1$ and $d_2$. Suppose further that $D_1$ and $D_2$ are ordered with the crossings outside of $B$ listed first and in the same order, followed by the crossings within $B$ in the order dictated by $T_1$ and $T_2$, respectively.
We will explicitly define maps $\hat{F}, \hat{G}, \tilde{h}_1$, and $\tilde{h}_2$ that give an equivalence between $C(D_1)$ and $C(D_2)$. First we will lay out a few definitions and notational conventions.

Throughout this proof $B^c$ will denote $\mathbb{R}^2 - B$. The restriction of a smoothing $M \in C(D_i)$ to $B$ will be referred to as $M_B$, while the restriction of $M$ to $B^c$ will be denoted $M_{B^c}$.

Every tangle cobordism $E$ originating at $M_B$ extends to a link cobordism $E^*$ originating at $M$, where $E^*$ consists of $E$ inside $B \times I$ and the identity on $M_{B^c}$ in $B^c \times I$. Extend this $\cdot^*$ operation linearly. If $E : M_B \to N_B$ and $F : N_B \to P_B$, then the equality $(FE)^* = F^*E^*$ follows directly from the definition. Note that this cobordism extension does depend on the full smoothing $M$ on which $E$ is to be extended, although the notation does not explicitly reference $M$. The $M$ in question should always be obvious in context.

We can lift a tangle map $H : C(T_1) \to C(T_2)$ in two ways. The first we will call a lift of $H$ and denote $\hat{H}$. The lift $\hat{H} : C(D_1) \to C(D_2)$ is defined as follows:

Consider a pair $M \in C(D_1)$ and $N \in C(D_2)$. The smoothings $M_B$ and $N_B$ are connected in $H$ by a cobordism $H_{N_B,M_B}$. Define a cobordism
\[ \tilde{H}_{N,M} = \begin{cases} (H_{N_B,M_B})^* & \text{if } M \cap B^c = N \cap B^c \\ 0 & \text{otherwise} \end{cases} \]

This \( \tilde{H}_{N,M} \) is the \( N, M \) entry in the matrix \( \tilde{H} \). Build all other entries in the matrix \( \tilde{H} \) according to the same rule.

Before defining our second lift operation, we must define a coefficient function on smoothings in \( C(D) \). Let

\[ a(M) = (-1)^k, \]

where \( k \) is equal to the number of crossings outside of \( B \) that are 1-resolved in the smoothing \( M \).

The **skew lift** \( \tilde{H} \) of a map \( H \) is defined in a similar fashion as the pure lift, except that the \( N, M \) entry is given by

\[ \tilde{H}_{N,M} = \begin{cases} a(M)(H_{N_B,M_B})^* & \text{if } M \cap B^c = N \cap B^c \\ 0 & \text{otherwise} \end{cases} \]

Both \( \hat{H} \) and \( \tilde{H} \) are additive, meaning that both \( \hat{H} + G = \hat{H} + \hat{G} \) and \( \tilde{H} + G = \tilde{H} + \tilde{G} \). It also happens that

\[ \tilde{H}_{N,M} = a(M)\hat{H}_{N,M}. \]
We claim that $\hat{F}$ and $\hat{G}$ are chain maps and that together with $\tilde{h}_i$ they define a chain homotopy equivalence between $C(D_1)$ and $C(D_2)$. First we will show that $\hat{F}$ is a chain map.

Before we can proceed we must describe how to decompose the differentials $\overline{d}_i$ from $C(D_i)$ in a useful way. The diagrams $D_i - T_i$ are tangles in their own right, so each has a cube $C(D_i - T_i)$ with differential that we will denote $d^c_i$. This differential can also be lifted to a $\hat{d}^c_i$. Note that in this case the lift occurs inside of $B$, meaning that $d^c_i$ is extended by the appropriate identities inside of $B$. Evaluating each side on generators confirms that $\overline{d}_i$ decomposes as

$$\overline{d}_i = \hat{d}^c_i + \tilde{d}_i.$$

In order to confirm that $\hat{F}$ is a chain map we need to check that

$$\overline{d}_2 \hat{F} = \hat{F} \overline{d}_1.$$

Decomposing $\overline{d}_i$ as above yields

$$(\hat{d}^c_2 + \tilde{d}_2) \hat{F} = \hat{F} (\hat{d}^c_1 + \tilde{d}_1).$$

This can be proven through the following pair of equations:

$$\tilde{d}_2 \hat{F} = \hat{F} \tilde{d}_1$$

$$\hat{d}^c_2 \hat{F} = \hat{F} \hat{d}^c_1.$$
To prove that each holds, we must show that the matrices on the left and right agree in each of their entries. Let $M \in C^i(D_1)$ and $N \in C^{i+1}(D_2)$. Then the $N,M$ entry in $\tilde{d}_2F$ is given by

$$
(\tilde{d}_2F)_{N,M} = \sum_{P \in C^i(D_2)} (\tilde{d}_2)_{N,P} \hat{F}_{P,M}
$$

$$
= \sum_{P \in C^i(D_2)} a(P)((d_2)_{N_B,P_B})^*(F_{P_B,M_B})^*
$$

$$
= \sum_{P \in C^i(D_2)} a(M)((d_2)_{N_B,P_B})^*(F_{P_B,M_B})^*
$$

$$
= a(M)(\sum_{P \in C^i(D_2)} (d_2)_{N_B,P_B} F_{P_B,M_B})^*
$$

$$
= a(M)(\sum_{Q \in C^{i+1}(D_1)} F_{N_B,Q_B}(d_1)_{Q_B,M_B})^*
$$

$$
= \sum_{Q \in C^{i+1}(D_1)} (F_{N_B,Q_B})^*a(M)((d_1)_{Q_B,M_B})^*
$$

$$
= \sum_{Q \in C^{i+1}(D_1)} \hat{F}_{N,Q}(\tilde{d}_1)_{Q,M}
$$

$$
= \hat{F}d_1N,M.
$$

The fifth equation in this sequence holds because the tangle map $F$ is a chain map, so $Fd_1 = d_2F$. All other steps are simple applications of the linear and multiplicative properties of $\cdot^*$ and the fact that $a(P) = a(M)$ whenever $P \cap B^c = M \cap B^c$. Note that if $M_{B^c} \neq N_{B^c}$ then this entire sum is zero.
A similar argument shows that $\hat{d}_2 \hat{F} = \hat{F} \hat{d}_1$. Suppose that $M \in C^i(D_1)$ and $N \in C^{i+1}(D_2)$. Then the $N, M$ entry in $\hat{d}_2 \hat{F}$ is

$$ (\hat{d}_2 \hat{F})_{N,M} = \sum_{P \in C^i(D_2)} (\hat{d}_2^c)_{N,P} \hat{F}_{P,M} $$

$$ = \sum_{Q \in C^{i+1}(D_1)} \hat{F}_{N,Q} (\hat{d}_1^c)_{Q,M} $$

$$ = (\hat{F} \hat{d}_1^c)_{N,M}. $$

This holds because $(\hat{d}_2^c)_{N,P}$ and $\hat{F}_{P,M}$ are cobordisms that have their non-trivial portions outside if $B$ and inside of $B$, respectively, implying that a time-reordering isotopy takes one composition to the other.

A symmetric argument proves that $\hat{G}$ is also a chain map on $C(D_2)$.

Now we need to confirm that the chain homotopy equivalence relations hold. Consider that

$$ \tilde{d}_1 \tilde{h}_1 + \tilde{h}_1 \tilde{d}_1 = (\hat{d}_1^c + \tilde{d}_1) \tilde{h}_1 + \tilde{h}_1 (\hat{d}_1^c + \tilde{d}_1) $$

$$ = \hat{d}_1^c \tilde{h}_1 + \tilde{d}_1 \tilde{h}_1 + \tilde{h}_1 \hat{d}_1^c + \tilde{h}_1 \tilde{d}_1 $$

$$ = \tilde{d}_1 \tilde{h}_1 + \tilde{h}_1 \tilde{d}_1. $$

Notice that the first and third terms in this second line cancel. To see why, consider that for $M, N \in C^i(D_1)$, the following holds:
\[(\widehat{d_1} \widehat{h_1})_{N,M} = \sum_{P \in C^{i-1}(D_1)} (\widehat{d_1})_{N,P} (\widehat{h_1})_{P,M} \]
\[= \sum_{P \in C^{i-1}(D_1)} (\widehat{d_1})_{N,P} a(M)(\widehat{h_1})_{P,M} \]
\[= \sum_{Q \in C^{i+1}(D_1)} a(M)(\widehat{h_1})_{N,Q}(\widehat{d_1}_{Q,M}) \]
\[= \sum_{Q \in C^{i+1}(D_1)} (-1)a(Q)(\widehat{h_1})_{N,Q}(\widehat{d_1}_{Q,M}) \]
\[= (-1) \sum_{Q \in C^{i+1}(D_1)} a(Q)(\widehat{h_1})_{N,Q}(\widehat{d_1}_{Q,M}) \]
\[= (-1) \sum_{Q \in C^{i+1}(D_1)} (\widehat{h_1})_{N,Q}(\widehat{d_1}_{Q,M}) \]
\[= (-1)(\widehat{h_1} \widehat{d_1})_{N,M}. \]

This uses several substitutions that warrant explanation. Since the \(Q\) at the top of each \((\widehat{d_1})_{Q,M}\) differs from \(M\) in exactly one resolution outside of \(B\), each \(a(Q) = -a(M)\). We also use the fact that \((\widehat{d_1})_{N,P}\) and \((\widehat{h_1})_{P,M}\) commute, which holds because their non-trivial portions are isolated, meaning they can be time re-ordered.

All that remains is to show that

\[\widehat{d_1} \widehat{h_1} + \widehat{h_1} \widehat{d_1} = \widehat{G}\widehat{F} - I,\]

which relies on the fact that \(d_1 h_1 + h_1 d_1 = GF - I\).
\[
(\tilde{d}_1\tilde{h}_1 + \tilde{h}_1\tilde{d}_1)_{N,M} = \sum_{P \in C^{i-1}(D_1)} (\tilde{d}_1)_N P(\tilde{h}_1)_{P,M} + \sum_{P' \in C^{i+1}(D_1)} (\tilde{h}_1)_N P'(\tilde{d}_1)_{P',M}
\]

\[
= \sum_{P \in C^{i-1}(D_1)} (\tilde{d}_1)_N P(\tilde{h}_1)_{P,M} + \sum_{P' \in C^{i+1}(D_1)} (\tilde{h}_1)_N P'(\tilde{d}_1)_{P',M}
\]

\[
= \sum_{P \in C^{i-1}(D_1)} ((d_1)_{N_B,P_B})^*((h_1)_{P_B,M_B})^*
\]

\[
+ \sum_{P' \in C^{i+1}(D_1)} ((h_1)_{N_B,P_B})^*((d_1)_{P_B,M_B})^*
\]

\[
= \left( \sum_{P \in C^{i-1}(D_1), P \cap B^c = M \cap B^c} (d_1)_{N_B,P_B} (h_1)_{P_B,M_B} \right)^*
\]

\[
+ \sum_{P' \in C^{i+1}(D_1), P' \cap B^c = M \cap B^c} (h_1)_{N_B,P_B} ((d_1)_{P_B,M_B})^*
\]

\[
= \left( \sum_{Q \in C^i(D_2), Q \cap B^c = M \cap B^c} G_{N_B,Q_B} F_{Q_B,M_B} - I_{N_B,M_B} \right)^*
\]

\[
= \sum_{Q \in C^i(D_2), Q \cap B^c = M \cap B^c} (G_{N_B,Q_B})^* (F_{Q_B,M_B})^* - (I_{N_B,M_B})^*
\]

\[
= \sum_{Q \in C^i(D_2)} (\hat{G})_{N,Q}(\hat{F})_Q,M - \hat{I}_{N,M}
\]

\[
= (\hat{G} \hat{F} - \hat{I})_{N,M}.
\]
Thus \( \hat{d}_1 \hat{h}_1 + \hat{h}_1 \hat{d}_1 = \hat{G} \hat{F} - I \). A symmetric argument shows that
\( \hat{d}_2 \hat{h}_2 + \hat{h}_2 \hat{d}_2 = \hat{F} \hat{G} - I \), so we conclude that \( C(D_1) \) and \( C(D_2) \) are
chain homotopy equivalent.

\[ \square \]

As an example application of Theorem 2, consider lifting the simplest
non-trivial map available: the chain map of tangles shown in Figure 5.

![Figure 5. Lift F.](image)

Suppose that the tangle on the left is contained within a link diagram
\( D \). Lifting this \( F \) to a chain map on \( C(D) \) yields the saddle map
described in Example 1.
3. Invariance

We have now developed all the machinery needed to formulate our invariance theorem.

**Theorem 3** (Bar-Natan [1]). *Let $D$ be a diagram for an oriented link $L$. The chain homotopy equivalence class of $C(D)$ is an invariant of $L$.*

Theorem 2 allows us to focus solely on the various versions of the Reidemeister moves. We will show that each has an associated chain homotopy equivalence between the cubes for its before and after tangles. These equivalences will be expressed here in full detail. We will also need to address the changes in $C(D)$ that arise when a different order is chosen on the crossings in $D$.

$$R1 : \begin{array}{c}
\quad \uparrow \\
\quad \downarrow \\
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
\quad \downarrow \\
\quad \uparrow \\
\end{array}$$

3.1. **Invariance Under R1.** The left handed Reidemeister 1 move induces a homotopy equivalence between complexes that is explicitly given by the maps $F$ and $G$ in Figure 6. The maps shown are technically $1 \times 1$ matrices, but the matrix notation will be dropped here. In order to confirm that $F$ and $G$ actually define an equivalence, we must check that they are both chain maps (that the maps $F$, $G$, and $d$ commute.
in the diagram in Figure 6) and that both compositions $FG$ and $GF$ are homotopic to the appropriate identity maps.

![Diagram](image)

**Figure 6.** The chain maps induced by the negative Reidemeister twist/untwist.

The only non-trivial commutative square that needs to be confirmed is that $Fd = 0$. This holds since the two terms in $Fd$ are isotopic and differ in sign.

The composition $FG$ is actually equivalent to the identity via the (T) relation followed by an isotopy, as shown in Figure 7.

The fact that $GF$ is homotopic to $I$ can be seen in Figure 8, which holds because it is a rearrangement of the (4t) relation, where one $B^2$
Figure 7. The composition $FG$ is equivalent to the identity. is removed from each of the left and top components and two from the bottom component, as shown.

Figure 8. The composition $GF - I = dh$ by the (4t) relation.

For a diagram defining the necessary maps for the right handed twist/untwist move see page 105 in Appendix A.
3.2. **Invariance Under R2.** The Reidemeister 2 tuck/untuck move induces an equivalence of cubes explicitly defined in Figure 9. The maps $F$, $G$, and $h$ are shown.

![Diagram](image-url)

*Figure 9. The chain maps associated with $R2$.*

Again, we first need to confirm that these are actually chain maps. The non-trivial calculations that need to be checked are that $F_0 d_{-1} = 0$ and $d_0 G = 0$. These equations hold because
Now we must check that $F$ and $G$ give a chain homotopy equivalence.

It turns out that we can make an even stronger statement. The map $F$ is actually an example of a deformation retract, meaning that $FG = I$, $GF - I = dh + hd$, and $hG = 0$. We will use this fact later when we present the maps induced by the third Reidemeister move.

To prove the first and third conditions we make use of the $(S)$ relation, as shown in Figure 10.

$$F_0d_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$d_0G = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$F_0G_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$h_0G_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$
To finish the proof we need only show that $GF = I + dh + hd$.

Consider the two sides of this equation separately, with the right side broken up into manageable pieces as shown in Figure 11.

$$d_{-1}h_0 = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ \vdots \end{pmatrix} \end{pmatrix} & \begin{pmatrix} \begin{pmatrix} 0 \\ \vdots \end{pmatrix} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$h_1d_0 = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ \vdots \end{pmatrix} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$d_{-1}h_0 + h_1d_0 + I = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

$$G_0F_0 = \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

**Figure 11.** $GF$ and $d_{-1}h_0 + h_1d_0 + I$ in pieces.
The first three entries of $GF$ and $d_1 h_0 + h_1 d_0 + I$ are clearly equivalent. It remains to confirm the equivalence of the fourth entries, which holds because of an easy application of the $(4t)$ relation illustrated in the figure above.
3.3. Invariance Under R3. In order to explicitly determine the chain maps induced by the third Reidemeister move, we again follow Bar-Natan’s approach, using the homotopy equivalence of certain mapping cones to determine the induced cobordisms. In [1], Bar-Natan presents chain homotopy equivalence maps induced by one version of the third Reidemeister move and leaves the other as an exercise. We will complete the second version in detail here. We begin with a general construction.

\[
\begin{array}{ccccccc}
C^{n-1} & \xrightarrow{c_{n-1}} & C^n & \xrightarrow{c_n} & C^{n+1} & \xrightarrow{c_{n+1}} & \ldots \\
\Psi_{n-1} & & \Psi_n & & \Psi_{n+1} & & \\
D^{n-1} & \xrightarrow{d_{n-1}} & D^n & \xrightarrow{d_n} & D^{n+1} & \xrightarrow{d_{n+1}} & \ldots \\
G_{n-1} & \xrightarrow{F_{n-1}} & G_n & \xrightarrow{F_n} & G_{n+1} & \xrightarrow{F_{n+1}} & \\
E^{n-1} & \xrightarrow{e_{n-1}} & E^n & \xrightarrow{e_n} & E^{n+1} & \xrightarrow{e_{n+1}} & \ldots \\
\end{array}
\]

**Figure 12.** A chain map and a deformation retract.

Given a chain map \( \Psi : C \to D \) and a deformation retract \( G : D \to E \) (with corresponding inclusion \( F \) and homotopy \( h \)) as shown, Lemma 5
in [1] tells us that the cones $\Gamma(\Psi)$ and $\Gamma(G\Psi)$ are chain homotopy equivalent.

The induced chain maps $\overline{G}$ and $\overline{F}$ and the homotopy $\overline{h}$ in this equivalence are defined in Figure 13.

$$
\begin{align*}
\overline{d}_n &= \begin{pmatrix} -c_n & 0 \\ \Psi_n & d_{n-1} \end{pmatrix} \\
\overline{h}_{n+1} &= \begin{pmatrix} 0 & 0 \\ -h_n h_{n+1} \Psi_{n+1} & h_n \end{pmatrix} \\
\overline{e}_n &= \begin{pmatrix} -c_n & 0 \\ G_n \Psi_n & e_{n-1} \end{pmatrix} \\
\overline{G}_n &= \begin{pmatrix} I & 0 \\ -G_{n-1} h_n \Psi_n & G_{n-1} \end{pmatrix} \\
\overline{F}_n &= \begin{pmatrix} I & 0 \\ h_n \Psi_n & F_{n-1} \end{pmatrix} \\
\overline{G}_{n+1} &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \\
\overline{F}_{n+1} &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}
\end{align*}
$$

**Figure 13.** The induced maps between $\Gamma(\Psi)$ and $\Gamma(G\Psi)$.

To prove that $\overline{G}$ and $\overline{F}$ actually define a chain homotopy equivalence, we need to first check that they are chain maps, i.e. that the squares in Figure 13 commute. Recall that the following conditions hold for $F, G,$ and $h$ and for all $n$:

\[
\begin{align*}
F_n G_n - I &= d_{n-1} h_n + h_{n+1} d_n \\
G_n F_n &= I \\
h_n F_n &= 0 \\
e_n G_n &= G_{n+1} d_n \\
F_{n+1} e_n &= d_n F_n
\end{align*}
\]
Using this set of equalities we get

\[ F_{n+1} e_n = \begin{pmatrix} I & 0 \\ h_{n+1} \Psi_{n+1} & F_n \end{pmatrix} \begin{pmatrix} -c_n & 0 \\ G_n \Psi_n & e_{n-1} \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ -h_{n+1} \Psi_{n+1} c_n + F_n G_n \Psi_n & F_n e_{n-1} \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ -h_{n+1} d_n \Psi_n + F_n G_n \Psi_n & d_{n-1} F_n \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ (h_{n+1} d_n + F_n G_n) \Psi_n & d_{n-1} F_n \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ (I + d_{n-1} h_n) \Psi_n & d_{n-1} F_n \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ \Psi_n & d_{n-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ h_n \Psi_n & F_n \end{pmatrix} \]

\[ = \overline{d}_n F_n \]

and also

\[ \overline{a}_{n+1} \overline{d}_n = \begin{pmatrix} I \\ -G_n h_{n+1} \Psi_{n+1} + G_n \Psi_n \end{pmatrix} \begin{pmatrix} -c_n & 0 \\ \Psi_n & d_{n-1} \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ G_n \Psi_n + G_n h_{n+1} \Psi_{n+1} c_n & G_n d_{n-1} \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ G_n \Psi_n + G_n h_{n+1} d_n \Psi_n & G_n d_{n-1} \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ G_n (I + h_{n+1} d_n) \Psi_n & G_n d_{n-1} \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ G_n (F_n G_n - d_{n-1} h_n) \Psi_n & G_n d_{n-1} \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ G_n F_n G_n \Psi_n - G_n d_{n-1} h_n \Psi_n & e_{n-1} G_n \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ G_n \Psi_n - e_{n-1} G_n d_{n-1} h_n \Psi_n & e_{n-1} G_n \end{pmatrix} \]

\[ = \begin{pmatrix} -c_n & 0 \\ G_n \Psi_n & e_{n-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -G_n h_n \Psi_n & G_n \end{pmatrix} \]

\[ = \overline{e}_n \overline{g}_n. \]
Next we must check that $FG$ and $GF$ are each homotopic to the appropriate identities.

\[
G_nF_n = \begin{pmatrix}
I & 0 \\
-G_{n-1}h_n\Psi_n & G_{n-1}
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
h_n\Psi_n & F_{n-1}
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
-G_{n-1}h_n\Psi_n + G_{n-1}h_n\Psi_n & G_{n-1}F_{n-1}
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}.
\]

It remains to confirm that $F_nG_n - I = \overline{d}_{n-1}h_n + \overline{h}_{n+1}\overline{d}_n$. Consider the two sides separately.

\[
F_nG_n - I = \begin{pmatrix}
I & 0 \\
h_n\Psi_n & F_{n-1}
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
-G_{n-1}h_n\Psi_n & G_{n-1}
\end{pmatrix} - \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
h_n\Psi_n - F_{n-1}G_{n-1}h_n\Psi_n & F_{n-1}G_{n-1} - I
\end{pmatrix}.
\]

On the other hand we have

\[
\overline{d}_{n-1}h_n + \overline{h}_{n+1}\overline{d}_n = \begin{pmatrix}
-c_{n-1} & 0 \\
\Psi_{n-1} & d_{n-2}
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
-h_{n-1}h_n\Psi_n & h_{n-1}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 \\
-h_nh_{n+1}\Psi_{n+1} & h_n
\end{pmatrix}
\begin{pmatrix}
-c_n & 0 \\
\Psi_n & d_{n-1}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
-d_{n-2}h_{n-1}h_n\Psi_n & d_{n-2}h_{n-1}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 \\
h_n\Psi_n + h_nh_{n+1}\Psi_{n+1}c_n & hNd_{n-1}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
-d_{n-2}h_{n-1}h_n\Psi_n + h_n\Psi_n + h_nh_{n+1}\Psi_{n+1}c_n & d_{n-2}h_{n-1} + hNd_{n-1}
\end{pmatrix}.
\]

The fourth entries in these two matrices are equal because $F$, $G$, and $h$ satisfy $F_nG_n - I = d_{n-1}h_n + h_{n+1}d_n$ for all $n$, while the third entries agree as seen by
Thus $\overline{F}$ and $\overline{G}$ define a chain homotopy equivalence between $\Gamma(\Psi)$ and $\Gamma(F\Psi)$. \hfill \Box

We will now use the maps presented in Figure 13 to specify the chain homotopy equivalence induced by the third Reidemeister move. Let $\Psi$ and $\Psi'$ be the chain maps shown below. Each is the map induced by a saddle in the center its respective diagram (see Example 1).

$$\Psi : C\left(\begin{array}{c} \includegraphics{Psi_diagram} \end{array}\right) \longrightarrow C\left(\begin{array}{c} \includegraphics{Psi_prime_diagram} \end{array}\right) \quad \Psi' : C\left(\begin{array}{c} \includegraphics{Psi_prime_diagram} \end{array}\right) \longrightarrow C\left(\begin{array}{c} \includegraphics{Psi_diagram} \end{array}\right)$$

Notice that the co-domains of $\Psi$ and $\Psi'$ both deformation retract, via the Reidemeister 2 untuck map presented in Figure 9, onto the same crossingless tangle. Call these retracts $G$ and $G'$, respectively, with corresponding inclusions $F$ and $F'$ and homotopy maps $h$ and $h'$.

Observe as well that

$$C\left(\begin{array}{c} \includegraphics{Psi_diagram} \end{array}\right) = \Gamma(\Psi) \quad \text{and} \quad C\left(\begin{array}{c} \includegraphics{Psi_prime_diagram} \end{array}\right) = \Gamma(\Psi').$$
The retraction $G$ followed by the inclusion $F$ together induce a chain homotopy equivalence $F'G : \Gamma(\Psi) \to \Gamma(\Psi')$, which is given by

$$
\begin{pmatrix}
I & 0 \\
h'\Psi' & F'
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
-Gh\Psi & G
\end{pmatrix}
= 
\begin{pmatrix}
I & 0 \\
h'\Psi'I - F'IGh\Psi & F'IG
\end{pmatrix}.
$$

These maps are all shown in Figure 14. Unlabeled arrows are either identity cobordisms or differential cobordisms, as appropriate. Arrows marked with only negative signs are differential cobordisms with negative coefficients. Omitted arrows are zero cobordisms.

All other arrows are marked with a smoothing picture that contains some combination of cup, saddle, and cap markers indicating what that cobordism does to the underlying smoothing. The exception to this is the monkey, which represents a cap followed by a monkey saddle and then a cup.

For a diagram explicitly defining the other version of the third Reidemeister move, see page 110.
Figure 14. A retraction $F$ followed by an inclusion $G$. 
Figure 15. The map induced by the third Reidemeister move.
3.4. **Re-ordering the Crossings.** We have shown that the chain homotopy equivalence class of $C(D)$ is invariant under the Reidemeister moves. We have not addressed, however, the impact on $C(D)$ of our choice for the ordering of the crossings in $D$. We will answer that question here by describing the chain homotopy equivalence induced by a re-ordering of the crossings in $D$.

Since any permutation of the crossing order can be decomposed as a sequence of transpositions, we need only describe the chain map associated to the transposition of two crossings in the ordering list. Suppose that diagrams $D$ and $D'$ agree everywhere except for the order of their crossings. In particular, suppose that the $i^{th}$ and $j^{th}$ crossings in $D$ have been reversed in $D'$. We will define chain maps $P^i$ between $C^i(D)$ and $C^i(D')$ induced by this transposition.

Set $I_M$ to be the identity cobordism $M \times I \subset \mathbb{R}^2 \times I$. Define $b_M = -1$ if the $i^{th}$ and $j^{th}$ crossings have both been 1-resolved in $M$. Otherwise let $b_M = 1$. Then for $M \in C^i(D_1)$ and $N \in C^i(D_2)$, define the $N,M$ entry of $P^i$ by

$$P^i_{N,M} = \begin{cases} 
    b_M I_M & \text{if } M = N \\
    0 & \text{otherwise}
\end{cases}$$

In order to prove that $P$ defines a chain homotopy equivalence between $C(D_1)$ and $C(D_2)$, we must first show that $P$ commutes $d_1$ and $d_2$. To
do this we will use the fact that for smoothings \( M \in C^i(D_1) \) and \( N \in C^{i+1}(D_2) \),

\[
(d_1)_{N,M} = (d_2)_{N,M} \quad \text{if and only if} \quad b_M = b_N.
\]

We can compute the \( N, M \) entry in \( d_2 P \) to be

\[
(d_2 P)_{N,M} = \sum_{Q \in C^i(D_2)} (d_2)_{N,Q} P_{Q,M} = (d_2)_{N,M} b_M P_{M} = (d_2)_{N,M} b_M = b_N (d_1)_{N,M} = b_N I_N (d_1)_{N,M} = \sum_{R \in C^{i+1}(D_1)} P_{N,R} (d_1)_{R,M} = (P d_1)_{N,M}
\]

Thus \( P \) is a chain map. Define the chain map \( P' : C(D_2) \to C(D_1) \) using a definition symmetric to that given for \( P \). Then \( PP' = I \) and \( P'P = I \), meaning that \( P \) and \( P' \) define a chain homotopy equivalence between \( C(D_1) \) and \( C(D_2) \).

Note that this argument does not rely on Theorem 2. We had to avoid applying this theorem here because in the process of proving 2 we made assumptions regarding the order of the crossings in our cubes.
Now that we have presented a general formula for $P$, however, we can describe the same $P$ by lifting a simple tangle map. Consider the chain map of cubes presented in Figure 16, where the two tangles are assumed to lie in diagrams that differ only in the order of these two crossings. All horizontal arrows represent identity cobordisms.

\[ C(\begin{array}{c}
\includegraphics{tangle1}
\end{array}) \quad C(\begin{array}{c}
\includegraphics{tangle2}
\end{array}) \]

Figure 16. The chain map induced by a transposition of crossings in the order list.
This is easily confirmed to be a chain homotopy equivalence between the tangle cubes shown. Theorem 2 allows us to lift this tangle cube equivalence to a chain homotopy equivalence between $C(D)$ and $C(D')$, which is the same chain map defined above.
4. Recovering Khovanov’s Theory

In Section 2 we saw that in order to force $C(D)$ to yield an invariant of $L$ we have to impose a set of local relations on the cobordisms in $C(D)$. In this section we will try to make sense of these relations and their implications. In particular, we will see that if 2 is invertible in $R$, the $R$-modules $V_M$ simplify considerably under the $(S), (T),$ and $(4t)$ relations. We will see how imposing one more relation (the $(3g)$ relation) on cobordisms in the cube of resolutions effectively reduces $C(D)$ to the Khovanov chain complex as originally defined in [6]. In particular, we will prove:

**Theorem 4.** Suppose $\Sigma$ is an orientable surface with boundary $M$. As an element in $V_M$, $\Sigma$ is equivalent to a linear combination of surfaces that each satisfy two conditions: All closed components are of genus 3 and all boundary components are once punctured spheres ($S^*$) or once punctured tori ($T^*$).

The relation $(4t)$ implies a second relation $(N)$, called the neck cutting relation, which will simplify the conversation to come. Consider the punctured surface shown to the left in the following diagram.
The (4t) relation implies the equation shown. After dividing both sides by 2, we have the (N) relation below. We can now use (N) to cut compressible necks whenever we find doing so convenient.

\[ \text{(N)} \quad \begin{array}{c} \hspace{2em} \quad = \frac{1}{2} \quad \end{array} \]

**Proof of Theorem 4:** Let Σ be a cobordism in \( V_M \). We can trim the complicated portions of Σ away from \( M \) in a methodical way.

\[ \begin{array}{c} \hspace{2em} \quad = \frac{1}{2} \quad \end{array} \]

**Figure 17.** A boundary parallel neck to cut.

Let \( c \) be an innermost circle in \( M \). If \( c \) already bounds an \( S^\bullet \) or \( T^\bullet \), isotope that portion of Σ close to \( M \). If not, then there exists a cuttable neck in Σ that is parallel to the circle \( c \). Cut that neck and isotope the resulting boundary component on \( c \) out of the way in each
resulting summand. We can repeat this process for subsequent inner-
most circles that have not been trimmed away until what remains is a
linear combination of surfaces, each having only boundary components
that are $S^*$’s or $T^*$’s. All higher genus components are closed.

$$=\frac{1}{2} \left( \begin{array}{c}
\includegraphics{figure18a}
\end{array} \right)$$

$$=\begin{array}{c}
\includegraphics{figure18b}
\end{array}$$

**Figure 18.** Unclasping a pair of 1-handles.

Suppose that a surface contains a pair of compressible clasped 1-
handles. Pick one of these 1-handles and cut it with $N$. The resulting
sum contains two isotopic copies of a surface of the same genus as the
original, in which the handles are no longer clasped. This process can
be repeated if necessary. Thus any knotted closed surface is equivalent
under the relations in $l$ to a standardly embedded surface of the same
genus.

Denote by $\Sigma_g$ the standardly embedded closed surface of genus $g$ in
$S^3$. Cutting the neck in the two holed torus $\Sigma_2$ at the line shown and
applying the $(T)$ relation yields:
\[
\begin{align*}
\begin{array}{c}
\includegraphics{diagram1}
\end{array}
\end{align*}
\]

From this we can conclude that \( \Sigma_2 = 0 \), meaning any cobordism in \( C(D) \) containing a \( \Sigma_2 \) component is equivalent to the 0 cobordism.

\[
\begin{array}{c}
\includegraphics{diagram2}
\end{array}
\]

Similarly, if a closed surface of genus \( g > 3 \) is cut at the line shown in the diagram above, a sum of two closed surfaces is created. The first has a component of genus 2 and is therefore trivial. The other contains a \( \Sigma_{g-2} \). In the case \( g = 4 \), this second surface also contains a \( \Sigma_2 \) and is trivial. If \( g \geq 6 \) is even, then \( g - 2 > 3 \) is again even, so \( \Sigma_{g-2} \) is also trivial by induction. Therefore (N) and (T) imply that all surfaces containing closed components of even genus are trivial.

If instead \( g \geq 5 \) is odd, one summand contains a \( \Sigma_2 \) component and is therefore trivial, while the other contains a \( \Sigma_3 \) and a \( \Sigma_{g-2} \). This \( \Sigma_{g-2} \) is also of odd genus, so it can be further reduced by neck cutting. We can continue this reduction process until all remaining closed components are standardly embedded surfaces of genus three.
Therefore all surfaces are equivalent under (S), (T), and (4t) to a linear combination of surfaces with all closed components of genus exactly 3 and all boundary components either $S^\bullet$‘s or $T^\bullet$‘s.

If we extend our coefficient ring $R$ to include polynomials in a variable $t$, we can remove those genus three components with the $(3g)$ relation

$$ (3g) \quad \includegraphics{cobordism_3g} = \times 8t. $$

Every cobordism in $V_M$ is therefore equivalent, up to the relations in $l' = l \cup (3g)$, to an $R[t]$-linear combination of cobordisms in which each component is either a $T^\bullet$ or an $S^\bullet$. It will therefore be necessary to understand how the edge cobordisms in $C(D)$ act on these two generators.

Recall that any cobordism in $\mathbb{R}^2 \times I$ can be decomposed into a sequence of the four basic building blocks shown above. The $\cup$ cobordism is itself an $S^\bullet$, so we need only understand how the simple saddles and cap cobordisms $V$, $\Lambda$, and $\cap$ act on $S^\bullet$ and $T^\bullet$. Begin with the action of $\cap$. 
The first equation says that capping off a $T^\bullet$ is equivalent under $l'$ to multiplication by two. Capping off an $S^\bullet$ annihilates the cobordism.

Now consider the splitting saddle $V$ acting on $S^\bullet$ and $T^\bullet$.

The merging saddle $\Lambda$ acts on pairs in $\{S^\bullet, T^\bullet\}^2$. The four cases are illustrated below.
From these computations we conclude the following set of rules:

\[
\begin{align*}
CS^\bullet &= \times 0 & \Lambda(S^\bullet \sqcup S^\bullet) &= S^\bullet \\
CT^\bullet &= \times 2 & \Lambda(S^\bullet \sqcup T^\bullet) &= T^\bullet \\
VT^\bullet &= \frac{1}{2}T^\bullet \sqcup T^\bullet + \frac{i}{4}S^\bullet \sqcup S^\bullet & \Lambda(T^\bullet \sqcup S^\bullet) &= T^\bullet \\
VS^\bullet &= \frac{1}{2}T^\bullet \sqcup S^\bullet + \frac{1}{2}S^\bullet \sqcup T^\bullet & \Lambda(T^\bullet \sqcup T^\bullet) &= \frac{i}{2}S^\bullet
\end{align*}
\]

The topological system presented in \( C(D) \) has been effectively reduced by the relations in \( l' \) to a completely algebraic one. To formalize this transition from topology to algebra, define a functor \( \mathbb{F} \) that takes \( C(D) \) to a Frobenius algebra \( A \) generated by variables \( x \) and \( 1 \). In particular, define \( 2x = \mathbb{F}T^\bullet \) and \( 1 = \mathbb{F}S^\bullet \). The multiplication, comultiplication, birth and death maps in \( A \) are then defined by \( m = FA \), \( \Delta = \mathbb{F}V \), \( \iota = \mathbb{F}\cap \), and \( \epsilon = \mathbb{F}\cup \). The differential rules presented above
for $C(D)$ become:

$$
i 1 = 0 \quad m(1 \otimes 1) = 1
$$

$$
i x = 1 \quad m(1 \otimes x) = x
$$

$$
\Delta x = x \otimes x + t(1 \otimes 1) \quad m(x \otimes 1) = x
$$

$$
\Delta 1 = x \otimes 1 + 1 \otimes x \quad m(x \otimes x) = t(1)
$$

$$
\epsilon 1 = 1
$$

The functor $\mathbb{F}$ maps $C(D)$ to a chain complex of $R[t]$-modules in $A$. If we set $t = 0$ and choose $R = \mathbb{Z}$, we find that $\mathbb{F}C(D)$ is actually the original chain complex defined by Khovanov [6].

If instead we choose $t = 1$ and set $R = \mathbb{Q}$, the chain complex $\mathbb{F}C(D)$ is the version of Khovanov’s complex developed by Lee in [7].

4.1. Quantum Grading. Note that cobordisms in $C(D)$ are naturally graded by the Euler characteristic. Define $p(S) = \chi(S) + h(S)$ for each surface $S$. A quick check confirms that the differential $d$ in $C(D)$ preserves $p$. Thus $C(D)$ is actually a bi-graded chain complex. Although not strictly necessary, we can shift $p$ globally by a factor of $n_+ - n_-$, allowing us to define the quantum grading on $C(D)$ by

$$
q(S) = p(S) + n_+ - n_- = \chi(S) + h(S) + n_+ - n_-
$$

By letting $\mathbb{F}$ preserve $q$ we pass the quantum grading on to $\mathbb{F}C(D)$. Thus $\mathbb{F}C(D)$ is also a bi-graded chain complex, with the quantum
grading in this context also denoted $q$. Note that $\chi T^* = -1$ and $\chi S^* = 1$, and that $\chi$ is additive over unions. Therefore when computing $q$ for a generator $s \in \mathbb{F}C(D)$ the $\chi$ contribution is given by counting $x$’s and 1’s. In particular, define $c(s)$ to be the number of 1’s in $s$ minus the number of $x$’s in $s$. Then

$$q(s) = c(s) + h(s) + n_+ - n_-.$$

The global shift by $n_+ - n_-$ is required by Khovanov [6] to recover the Jones polynomial from $\mathbb{F}C(D)$. 
5. Computing $Kh(L)$

For the remainder of this paper we will work with $R = \mathbb{Q}$ coefficients and assume $t = 0$. We will use the notation $CKh(D) = FC(D)$ for the Khovanov chain complex, the differential of which will be denoted $d$. The homology groups derived from $CKh(D)$ will be denoted $Kh(L)$, with the homology class of a cycle $T$ denoted $[T]$. 

Note that while the differential in $CKh(D)$ is actually height increasing, meaning that we are technically discussing a co-homology theory, we will continue the habit established by Khovanov of dropping the co- from our language. The chain complex $CKh(D)$ will always have compact support, so this abuse of the language does not present any complications.

When describing generators of $CKh(D)$, it will be convenient to retain some of the topological information that is technically forgotten by the functor $F$. To that end, we will use the following definition for a simple state (generator) in $CKh(D)$.

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {\textbf{0}};
  \node (B) at (2,0) {1};
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (A);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \draw (0.5,0) -- (0.5,-0.5);
  \draw (1.5,0) -- (1.5,-0.5);
  \draw (2.5,0) -- (2.5,-0.5);
\end{tikzpicture}
\end{center}

\textbf{Definition 5.} A simple state $t \in CKh(D)$ is a diagram consisting of the following:
• A smoothing $M_t$ of $D$, which is a 1-manifold obtained from $D$ by resolving each crossing as either a 0- or 1-resolution, according to the rule above (keeping the cross marks to encode which resolution has been performed at each crossing).

• A label on each component in $M_t$ with an element from $\{1, x\}$.

Notice that with this crossing and label information included we can recover both the $h$ and $q$ gradings from $t$, allowing us to determine directly from the diagram where the generator $t$ sits in $CKh(D)$.

5.0.1. Calculating $Kh(\emptyset)$. Although it is not particularly illustrative, we will calculate $Kh(\emptyset)$ for later reference. Note that the link diagram $D = \emptyset$ has $n_+ = n_- = 0$ and admits only the empty smoothing $M_\emptyset$. The associated $V_{M_\emptyset}$ is the $\mathbb{Q}$-module generated by the unique empty cobordism spanning $M_\emptyset$, meaning $V_{M_\emptyset} = \mathbb{Q}$ in height and quantum grading 0. Thus $C(D)$ is non-trivial precisely at $C^{0,0}(\emptyset) = \mathbb{Q}$. Therefore $CKh(\emptyset)$ also consists of a single $\mathbb{Q}$ at $(0, 0)$, with trivial differentials. This means that $Kh(\emptyset) = \mathbb{Q}$ at $(0, 0)$ and zero elsewhere. In the usual grid presentation we have $Kh(\emptyset)$ given by

<table>
<thead>
<tr>
<th>$h = 0$</th>
<th>$q = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}$</td>
<td>$\mathbb{Q}$</td>
</tr>
</tbody>
</table>
5.1. **The Hopf Link.** In Figure 1 we presented the cube of resolutions of a diagram $D$ for the left handed Hopf link $L$. We will now use this picture to generate $CKh(D)$, from which we will calculate $Kh(L)$. Below is a diagram displaying the simple states that generate $CKh(D)$, presented without the second grading.

We must determine the effect of $d$ on each of these twelve simple states. We find that

\[
\begin{align*}
    d\left(\begin{array}{c}
        & \\
        & \\
\end{array}\right) &= 0 \\
    d\left(\begin{array}{c}
        & \\
        & \\
    \end{array}\right) &= d\left(\begin{array}{c}
        & \\
        & \\
    \end{array}\right) = 0 \\
    d\left(\begin{array}{c}
        & \\
        & \\
    \end{array}\right) &= d\left(\begin{array}{c}
        & \\
        & \\
    \end{array}\right) = 0 \\
    d\left(\begin{array}{c}
        & \\
        & \\
    \end{array}\right) &= 0 \\
    d\left(\begin{array}{c}
        & \\
        & \\
    \end{array}\right) &= 0 \\
    d\left(\begin{array}{c}
        & \\
        & \\
    \end{array}\right) &= 0
\end{align*}
\]
From this we can identify the cycles in $CKh(D)$ to be the states listed in Figure 19. Of these, the three states that have been crossed off are also boundaries, so they represent trivial homology classes in $Kh(L)$. Furthermore, the second and third states listed in height 0 represent the same class.

We have found that there are four cycles in $CKh(D)$ that are not boundaries, so the Khovanov homology for $L$ has four non-trivial groups. See Appendix B for a full presentation of the bi-graded $CKh(D)$ and $Kh(L)$. The usual presentation for $Kh(L)$ is

<table>
<thead>
<tr>
<th>$Kh_{\mathbb{Q}}(L)$</th>
<th>$q=-6$</th>
<th>$q=-4$</th>
<th>$q=-2$</th>
<th>$q=0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h=0$</td>
<td></td>
<td>$\mathbb{Q}$</td>
<td>$\mathbb{Q}$</td>
<td></td>
</tr>
<tr>
<td>$h=-2$</td>
<td>$\mathbb{Q}$</td>
<td>$\mathbb{Q}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. Graph Structures and Triviality Conditions

The crossing information contained in the simple states, as presented in Definition 5, will facilitate our discussion of cycles in $CKh(D)$. If we treat the circles in a smoothing $M$ as vertices and the crossing markers as edges and impose the obvious incidence and adjacency rules, we can view $M$ as a graph $\Gamma_M$. Note that $\Gamma_M$ is actually a fat graph of sorts, although we will not use the added incidence order or inside/outside information inherent in the diagram $M$.

While the smoothing $M$ and the graph $\Gamma_M$ are not actually the same object, we will often refer to the graph properties of $M$ (or even of a simple state $t$ based on $M$) without any risk of confusion. We will also at times refer to $\Gamma_t$, by which we mean $\Gamma_M$ where $M$ is the marked smoothing in the simple state $t$.

We can refine $\Gamma_M$ by partitioning its edges into subsets $E^+ = \{\text{edges that have been 0-resolved in } M, \text{ noted with hash marks}\}$ and $E^- = \{\text{edges that have been 1-resolved}\}$. These are said to be the positive and negative edge sets of $\Gamma_M$, respectively.

Any edge that is incident to only one vertex is said to be degenerate. A graph $\Gamma$ is said to be degenerate if it contains a degenerate edge.
A graph in which all closed paths are of even length is said to be an **even graph**, while a graph containing a closed path of odd length is said to be an **odd graph**.

The vertices we then clump into subsets \( V^+ = \{ \text{vertices that are incident to at least one positive edge} \} \) and \( V^- = \{ \text{vertices that are incident to at least one negative edge} \} \). We will describe vertices in \( V^+ \) and \( V^- \) as **positive vertices** and **negative vertices**, respectively.

![Figure 20](image.png)

**Figure 20.** The knot 6_2, a smoothing \( M \), and its subgraphs.

The positive and negative subgraphs \( \Gamma^+ = (V^+, E^+) \) and \( \Gamma^- = (V^-, E^-) \) of \( \Gamma_M \) will be of interest in future discussions.

Each edge \( e \in \Gamma_M \) has an associated smoothing \( M_e \) derived from \( M \) by reversing the resolution of \( D \) at the edge \( e \). Notice that the resulting smoothing has a graph \( \Gamma_{M_e} = (V', E') \) that differs from \( \Gamma_M = (V, E) \) only in that exactly one pair of circles in one are merged in the other, and the edge \( e \in E \) is replaced by an \( e' \) in \( E' \) of the opposite sign.

With these notational conventions, we can describe the edge differential \( d_e \) for an edge \( e \in E_t^+ \) as follows: for a simple state \( t \) defined
on a smoothing $M$, $d_e(t)$ is the sum of states given by labeling $M_e$ as dictated by the edge cobordism $E_{M,M_e}$ in $C(D)$ and the rules presented on Page 49.

We can also recover the sign of $d_e(t)$ (inherited from $E_{M,M_e}$) by counting the positive edges (which correspond to 1-resolutions) in $t$ that appear in the order list before the crossing corresponding to $e$. Define $\text{sgn}(e,t) = -1$ if this count is odd, otherwise $\text{sgn}(e,t) = 1$.

The general state $T \in CKh(D)$ will be of the form

$$T = \sum_{k \in I} n_k t_k,$$

where $n_k \in \mathbb{Q}$ and $t_k$ is a simple state (as defined on page 51) for each $k$ in some finite index set $I$. Often the $t_k$ will consist of several simple states on the same smoothing, with slightly different labels.

One such example that often occurs when $T$ is a cycle in $CKh(D)$ is illustrated in Figure 21. The circles labeled with letters (as opposed to $x$'s and 1's) form a $pQr$-chain, which is shorthand for a sum of simple states on the smoothing shown. For each letter in the chain there is a simple state in which that letter is replaced with a 1 and all other letters in the chain are replaced with $x$'s. The lower case letters are associated with terms with coefficient 1 while the upper case terms carry a coefficient of $-1$. 
Figure 21. A $pQrsT$-chain.

Unless explicitly stated otherwise, take a $pqr$-chain to have all terms of the same sign, while a $pQr$-chain has terms that alternate throughout the chain, regardless of the chain length. If more than one $pQr$-chain occurs in $T$, some other set of letters in sequence will be used.

6.1. **Cycles in $CKh(D)$**. At times it will be necessary to determine whether a given state $T$ is a cycle in $CKh(D)$. In principle this is easy. Simply confirm that $d(T) = 0$. We will often bypass this computation with the following proposition (see p. 73 for a description of the chain map $\Phi_S$).

**Proposition 6.** If $D$ is a diagram for $L = \partial S$ for some oriented surface $S$ and $\Phi_S(1) = T$, then $T$ is a cycle in $CKh(D)$.

*Proof:* Since $CKh^{0,1}(\emptyset) = 0$, we must have $d(1) = 0$. Since $\Phi_S$ is a chain map,

$$dT = d\Phi_S(1) = \Phi_S(d1) = \Phi_S(0) = 0.$$
6.2. **Boundaries in** $CKh(D)$. If, on the other hand, we need to determine whether or not a given cycle $T$ is a boundary in $CKh(D)$, the calculation becomes considerably more difficult. To begin this analysis, we need more refined language to describe the relationship between $d$ and $T$.

If there exists an $S \in CKh(D)$ with $dS = T$, then we say $d$ hits $T$ strongly or that $S$ hits $T$ strongly. A more general condition is if $\text{proj}_T(d(S)) \neq 0$ for some $S$, in which case we say that $d$ hits $T$ weakly or that $S$ hits $T$ weakly.

If $dS = T = \sum_{k \in I} n_k t_k$ for some collection of simple states $t_k$, then $d$ hits $t_k$ weakly for each $k$. This can be re-stated as the following useful result.

**Proposition 7.** If $T = \sum_{k \in I} n_k t_k$ and there exists some $k \in I$ for which $d$ does not hit $t_k$ weakly, then $T$ is not a boundary.

We can now present a few examples of states representing classes for which the question of triviality can be answered conclusively.

**Theorem 8.** Let $D$ be a diagram for a link $L$. If $t$ is a simple cycle in $CKh(D)$ for which $\Gamma_t = \Gamma_t^-$ is non-degenerate, even, and connected, then $[t] = 0$ in $Kh(L)$ if and only if two or more vertices in $t$ are labeled $x$. 
Proof: Let $t \in CKh(D)$ be a simple state given by a label on a smoothing $M$. Suppose that $\Gamma_M = \Gamma^{-}_M$ is non-degenerate, even, and connected. Let $t$ have height and quantum grading $h_t$ and $q_t$, respectively.

Suppose further that there are two vertices $v_0$ and $v_n$ in $t$ labelled $x$. Let $P = \{v_0, e_1, v_1, e_2, \ldots e_n, v_n\}$ be a path in $\Gamma_M$ between $v_0$ and $v_n$. We can assume that the circle $v_i$ is labeled 1 in $t$ for all $i \neq 0, n$. For each edge $e_i \in P$ with $i \neq n$, let $s_i$ be the state given by $M_{e_i}$ labeled with 1’s on all vertices except $v_n$, which is labeled $x$. Let $s_n$ be given
by $M_{e_n}$ labeled with 1’s everywhere except for the newly merged circle, which is labeled $x$. Then

$$d\left(\sum_{i=1}^{n} \text{sgn}(e'_i, s_i)(-1)^i s_i\right) = t.$$  

The cancellation that occurs here is illustrated in the diagram on the previous page. We conclude that $[t]$ is trivial in $Kh(L)$.

For the other half of the proof, consider $t$ with a single vertex $v_0$ labeled $x$. The fact that $\Gamma_M = \Gamma^-_M$ tells us that $M$ is the only smoothing in $C(D)$ at height $h_t$. The generators of $CKh^{u_h,t}(D)$ are therefore the simple states $M_v$ given by labeling $M$ with an $x$ on the vertex $v$ and 1’s elsewhere.

Since $\Gamma_M$ is non-degenerate, all smoothings at the height $h_t - 1$ can be obtained from $\Gamma_M$ by merging two adjacent circles in $M$. The generators of $CKh^{u_h,t-1}(D)$ are precisely the states $s_e$ formed by labeling all circles in each $M_e$ with 1’s.

We wish to show that no linear combination

$$S = \sum_{e \in E_M} a_e s_e,$$

with coefficients $a_e \in \mathbb{Q}$, can satisfy $d(S) = t$.

As shown in Figure 22, each $d(s_e) = \text{sgn}(e', s_e)(M_u + M_v)$, where $u$ and $v$ are the two vertices incident to the edge $e$ in $\Gamma_t$. The collection of coefficients $\{b_e\} = \{a_e \text{sgn}(e', s_e)\}$ can be interpreted as a weighting
Figure 22. The full differential on $s_e$. 

Define the weight of a vertex $v$ in a weighted graph to be the sum of the weights on the edges incident to $v$. Then if $d(S) = t$, each vertex $v$ other than $v_0$ must have weight 0, while the weight of vertex $v_0$ must be 1. This cannot happen, however, because of the following lemma:

**Lemma 9.** Suppose $\Gamma$ is an even weighted graph. Then $\Gamma$ cannot have exactly one vertex of non-zero weight.

*Proof of Lemma:* We will begin by defining a set of graph reduction operations for weighted graphs.

Suppose that a graph $\Gamma$ happens to contain an even cycle that includes an edge $e_j$. A new weighted graph $\Gamma'$ can be gotten from $\Gamma$ by modifying the original weights on the cycle as follows: first subtract $a_j$ from the weight on $e_j$. Continue around the cycle, alternately adding...
and subtracting \( a_j \) to the weights on the edges in sequence. This process will be referred to as weight toggling the cycle to zero the edge \( e_j \).

Note that since the original cycle in \( \Gamma \) is of even length, weight toggling is well defined and preserves vertex weights.

![Diagram](image)

**Figure 23.** Toggling a cycle to zero the edge \( e_j \).

If a weighted graph \( \Gamma \) contains an edge \( e \) for which \( a_e = 0 \), then \( e \) can be removed to create a new weighted graph \( \Gamma' \) that otherwise agrees with \( \Gamma \). This reduction from \( \Gamma \) to \( \Gamma' \) will be referred to as an edge removal of \( e \) from \( \Gamma \). Edge removal also preserves vertex weights.

Consider a weighted graph \( \Gamma \) that contains a vertex \( v \) of valency 0. This \( v \) can be removed to create a new weighted graph \( \Gamma' \) that otherwise agrees with \( \Gamma \). The reduction from \( \Gamma \) to \( \Gamma' \) will be referred to as vertex removal of \( v \) from \( \Gamma \). Vertex removal preserves the weights of the remaining vertices in \( \Gamma \).

Now suppose that \( \Gamma \) is an even weighted graph, with edges \( \{e_n\} \) labelled \( \{a_n\} \), respectively. Assume further that exactly one vertex in \( \Gamma \) has non-zero weight.
Starting with $\Gamma$, recursively apply the following two step process: if a cycle exists, pick an edge $e$ within it and toggle the weights in that cycle to zero $e$. Then remove $e$. This process can be repeated until the resulting graph is a tree or forest $\Gamma_f$. Since weight toggling and edge removal both preserve vertex sums, $\Gamma_f$ must also have exactly one vertex of non-zero weight.

Since $\Gamma_f$ is a forest, it must have at least one leaf $v$ with weight 0. The single edge $e$ incident to $v$ therefore has weight 0. Remove the edge $e$. The vertex $v$ now has valency 0 and can be removed as well. Recursively isolate and remove zero sum leaves in this fashion until $\Gamma_f$ has been reduced to a single edge and its two incident vertices. This simple graph has exactly one vertex of non-zero weight because each of its predecessors did. On the other hand, the two remaining vertices must both have weights equal to the weight on the remaining edge. We have therefore arrived at a contradiction, and the original graph $\Gamma$ could not have had exactly one vertex of non-zero weight. □

If $t$ is labeled with all 1’s, then no label on any $M_e$ results in an $s$ with $q(s) = q(t)$. Thus there cannot exist an $s$ that hits $t$ even weakly. □
The last paragraph in this proof combines with Theorem 7 to prove:

**Proposition 10.** If \( T \) is a state in \( CKh(D) \) containing a summand that is non-degenerate, purely negative, and labeled 1 everywhere, then \([T]\) is non-trivial in \( Kh(L) \).

Looking at the other extreme for \( \Gamma \), we find:

**Proposition 11.** If \( t \) is a simple cycle in \( CKh(D) \) for which \( \Gamma_t = \Gamma_t^+ \), then \([t]\) is non-trivial in \( Kh(L) \).

**Proof:** Since \( \Gamma_t \) has no negative edges, \( CKh(D) \) is trivial in the height below \( t \). \( \square \)

**Proposition 12.** If \( t \) is a simple cycle with a negative edge \( e \) connecting two vertices \( v_1 \) and \( v_2 \) that are both labelled \( x \) and that are not adjacent in \( \Gamma_D^+ \), then \([t]\) is trivial in \( Kh(L) \).

**Proof:** Suppose \( t \) is a labeling of a marked smoothing \( M \). Define \( t_e \) to be the simple state gotten by labeling \( M_e \) to match the label in \( t \), except on the circle that is the merge of \( v_1 \) and \( v_2 \). Label this circle \( x \). Then we get \( d(t_e) = d_e(t_e) = t \). The key here is that \( t \) is a simple cycle, so all positive edges in \( t \) connect circles labeled \( x \). Thus \( d_f(t_e) = 0 \) for all \( f \neq e' \) in \( E_{Me}^+ \). \( \square \)
6.3. **Disjoint Unions.** We now prove a useful result about the disjoint union of simple states. Every simple state \( t \in CKh(D_1 \sqcup D_2) \) decomposes as a union \( t = t_1 \sqcup t_2 \), where \( t_1 \) and \( t_2 \) are simple states in \( CKh(D_1) \) and \( CKh(D_2) \), respectively.

**Theorem 13.** Let \( t = t_1 \sqcup t_2 \) be a simple state in \( CKh(D_1 \sqcup D_2) \). Then

1. the state \( t \) is a cycle in \( CKh(D_1 \sqcup D_2) \) if and only if \( t_1 \) and \( t_2 \) are cycles in \( CKh(D_1) \) and \( CKh(D_2) \).

2. Given that \( t \) is a cycle in \( CKh(D_1 \sqcup D_2) \), \([t]\) is non-trivial if and only if both \([t_1]\) and \([t_2]\) are non-trivial in \( Kh(L_1) \) and \( Kh(L_2) \), respectively.

Induction on Theorem 13 brings us to the following:

**Corollary 14.** If \( t = \sqcup_{i=1}^{n} t_i \) is a cycle in \( CKh(\sqcup_i D_i) \), then \([t]\) is non-trivial in \( Kh(\sqcup_i L_i) \) if \([t_i]\) is non-trivial in \( Kh(L_i) \) for each \( i \).

*Proof of Theorem 13:* Marked and labeled smoothings can be viewed as traditional states in \( CKh(D) \) embellished with the picture of the manifold from which they came. From this viewpoint the disjoint union operation behaves like a tensor product. Given that, the following results simply recreate the usual statements regarding tensors.
The first part of this theorem follows immediately from the fact that

\[
d(t) = \sum_{e \in E_i^+} d_e(t) = \sum_{e \in E_{i_1}^+} d_e(t_1 \sqcup t_2) + \sum_{e \in E_{i_2}^+} d_e(t_1 \sqcup t_2)
\]

\[
= d_1(t_1) \sqcup t_2 + (-1)^{|E_{i_1}^+|} t_1 \sqcup d_2(t_2)
\]

and the observation that since \(d_2(t_2) \perp t_2\), we know that \(d_1(t_1) \sqcup t_2\) and \(t_1 \sqcup d_2(t_2)\) are linearly independent in \(CKh(D_1 \sqcup D_2)\).

To prove (2), we first assume \(t_1 \sqcup t_2\) is a cycle in \(CKh(D_1 \sqcup D_2)\) for which \([t_1]\) is trivial in \(Kh(L_1)\). This means that there exists some \(S_1 = \sum_i s_i\) with \(d_1(S_1) = t_1\) in \(CKh(L_1)\). Then the state \(S = \sum_i s_i \sqcup t_2\) has

\[
d(S) = \sum_i d(s_i \sqcup t_2)
\]

\[
= \sum_i d_1(s_i) \sqcup t_2 + (-1)^{|E_{i_1}^+|} s_i \sqcup d_2(t_2)
\]

\[
= \sum_i d_1(s_i) \sqcup t_2
\]

\[
= (\sum_i d_1(s_i)) \sqcup t_2
\]

\[
= t_1 \sqcup t_2.
\]

In order to prove the second part of (2) we observe that

\[
CKh(D_1 \sqcup D_2) = CKh(D_1) \otimes CKh(D_2).
\]
Thus the Künneth formula gives us the isomorphism

$$\bigoplus_{i,j} \text{Kh}^{i,j}(L_1) \otimes \text{Kh}^{q-i,h-j}(L_2) \cong \text{Kh}^{q,h}(L_1 \sqcup L_2).$$

This isomorphism sends $[t_1] \otimes [t_2]$ to $[t_1 \otimes t_2] \in \text{Kh}(L_1 \sqcup L_2)$. Since $[t_1]$ and $[t_2]$ are each non-trivial in their respective homologies, $[t_1 \otimes t_2] \neq 0$ in $\text{Kh}(L_1 \sqcup L_2)$. □

6.4. **Representation Techniques.** As the crossing number in $D$ increases, proving non-triviality for a state $T$ through direct calculation quickly becomes unreasonable. At times it will be useful to apply a chain map $F$ to $CKh(D)$ that maps to a simpler complex $CKh(D')$, where $D'$ is a diagram for some other link $L'$. We can then analyze the representation $T' = F(T)$ instead of analyzing $T$ directly. If we find $[T'], \neq 0$ in $\text{Kh}(L')$ then we know $[T]$ to be non-trivial in $\text{Kh}(L)$ as well.

Although any chain map can be used for this purpose, we will focus on examples that can be created using compositions of the saddle map presented on page 11 and the maps induced by the first Reidemeister move, as presented in Section 3. In the rest of this section we will spell out a few guidelines for choosing productive representing maps.

One approach that we can take is to try to eliminate crossings in $D$. Consider a local picture of a crossing $c$. Define a **positive trim** of $c$
to be the chain map $\tau_c^+$ induced by a saddle along the arc shown in Figure 24, followed by a Reidemeister untwist.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure24}
\caption{A positive trim $\tau_c^+$ of $c$ in $D$.}
\end{figure}

This chain map has the effect of removing the positive edge at the crossing $c$ from any simple state in which $c$ is 0-resolved, while annihilating any simple state in which $c$ has been 1-resolved.

The position of the saddle arc relative to the crossing in $D$ is crucial. Consider the other version of this move, $\tau_c^-$, which we will call a **negative trim** of $c$ in $D$. The effect of $\tau_c^-$ is illustrated in Figure 25.

When discussing a simple state $t$ we will often abuse the notation by using “a trim of edge $e$” to mean a trim of the crossing $c$ of which $e$ is a resolution.
We will begin by applying crossing trimming to simple states. One reasonable tactic is to recursively trim all positive edges in \( t \). The resulting trimmed state, which we will denote \( t_{\text{trim}} \), will be a disjoint union of purely negative states. Each of these components can be classified as seen in Theorem 8. If they happen to each be non-trivial in their respective homology groups, then Theorem 13 tells us that \( t' \) is non-trivial, and hence \( t \) is as well. Consider the example in Figure 26, where \( t_{\text{trim}} \) is non-trivial by this argument.

The condition that makes edge trimming useful in this example is that each connected component in \( \Gamma_t^- \) is even and contains only one vertex labelled \( x \). This guarantees that after trimming all positive edges,
the purely negative components that remain will each individually be non-trivial. We have effectively proven the statement:

**Theorem 15.** Let $t$ be a simple non-degenerate cycle in $CKh(D)$ for which every component in $\Gamma_i^-$ is even and contains at most one vertex labelled $x$. Then $[t]$ is non-trivial in $Kh(L)$.

While relatively easy to check, the hypotheses of Theorem 15 are rather restrictive. More general applications of edge trimming, while less easily prescribed, can be applied to a larger class of examples.

Consider a simple cycle $t \in CKh(D)$. Suppose that selective trimming (of the positive and/or negative variety) of crossings in $D$ results in a state $t' \in CKh(D')$ that is a disjoint union of states, each of which is known to be of non-trivial in its respective homology. Then $t$ is also non-trivial, by Corollary 14.

Often a cycle $T \in CKh(D)$ will have many summands with different underlying manifolds. The trim maps $\tau_c^\pm$ annihilate those summands of $T$ in which $c$ is resolved one way, while hopefully preserving those
summands in which $c$ has the opposite resolution. A carefully selected sequence of edge trims can reduce $T$ to a representative consisting of a single simple state that can be proven non-trivial in homology.

For an example in which this approach is applied to a specific class, see the discussion of knot 946 in Section 7.3. In this case, we apply a sequence of three positive crossing trims to annihilate all but one of the summands in the class that we wish to prove non-trivial.

6.4.1. 

**Throwing Orientations out the Window.** As mentioned in Example 1, the saddle portion of each trimming map will often be disorienting. Our choice of orientation on the post-trim diagram $D'$ will effect the height and $q$-grading of the trimmed class $T'$. For the purposes of representing our original class, however, the height and $q$-grading of $T'$ are irrelevant. All that matters for this application is that $[T']$ be non-trivial in $Kh(D')$, which in each of our examples can be determined directly from the graph structures and labels on the generators of $T$. Thus when applying these representation techniques for the purpose of proving the non-triviality of $T$, we can completely neglect the orientation issues that technically arise every time we apply a chain map induced by a saddle.
7. Khovanov-Jacobsson Classes

In his paper [6], Khovanov observed that an oriented link cobordism $S$ in $\mathbb{R}^3 \times I$ induces a chain map

$$\Phi_S : CKh(\partial_0 S) \to CKh(\partial_1 S).$$

This in turn induces a map

$$\overline{\Phi}_S : Kh(\partial_0 S) \to Kh(\partial_1 S),$$

which Khovanov conjectured to be an invariant of the surface $S$. This statement was proven by Jacobsson [5] as

**Theorem 16** (Jacobsson). Let $S$ be a compact, oriented link cobordism between $L_0$ and $L_1$ in $\mathbb{R}^3 \times I$. Then, up to sign, $\overline{\Phi}_S : Kh(L_0) \to Kh(L_1)$ is invariant under boundary preserving isotopy. Furthermore, $\overline{\Phi}_S$ is non-trivial.

Note that Theorem 16 was originally proven to hold for Khovanov homology defined with $\mathbb{Z}$ coefficients. Tensoring all complexes and maps in question with $\mathbb{Q}$ changes Jacobsson’s statement to the version that we use here.

To determine the chain map induced by a surface $S$, fix a projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ of the movie frames at each $i \in I$. Then position $S$ in $\mathbb{R}^3 \times I$ such that it decomposes as a stack of simple cobordisms $S_k$, each of
which is an identity cobordism outside of some cylinder $B^2 \times I$. Inside
this cylinder each $S_k$ contains either a saddle, cup, or cap cobordism, or
an isotopy that is recorded by the projection $p$ as a single Reidemeister
move. Each of these $S_k$ induce chain maps $\Phi_{S_k}$ on $CKh$ as explicitly
presented in Sections 2 and 3. Thus

$$\Phi_S = \Phi_{S_n} \circ \Phi_{S_{n-1}} \circ \cdots \Phi_{S_1}.$$ 

Theorem 16 says that the chain homotopy type of $\Phi_S$ is independent
of our choice of decomposition. The proof of this theorem involves an
exhaustive check that each version of each of the fifteen elementary
movie moves presented by Carter and Saito [2] induce trivial homo-
morphisms (up to sign) on $Kh$.

One interesting property of the chain map $\Phi_S$ induced by a surface $S$
is that it shifts the quantum grading $q$ in a predictable way. The map
induced by a simple saddle shifts $q$ by $-1$, while the maps induced by
caps and cups shift $q$ by 1. Each of the Reidemeister maps preserve $q$.
Thus the map $\Phi_S$ shifts $q$ by $\chi S$.

7.1. **Closed Surfaces.** The (almost-)functoriality of Khovanov’s ho-
mology theory directly gives rise to an invariant of closed surfaces in
$B^4$, as follows: A closed oriented surface $S \subset B^4$ can be thought of
as a cobordism between empty links, $S : \emptyset \to \emptyset$, which induces a homomorphism $\overline{\Phi}_S$ on the level of homology. Since $Kh(\emptyset) = \mathbb{Q}$, this is $\overline{\Phi}_S : \mathbb{Q} \to \mathbb{Q}$. Jacobsson showed in Theorem 16 that the image of $1 \in \mathbb{Q}$ under $\overline{\Phi}_S$ is an invariant of $S$ under boundary preserving isotopy. This has since become known as the **Khovanov-Jacobsson number** of the surface $S$ and is denoted $KJ(S)$. Put more succinctly, $KJ(S) = |\overline{\Phi}_S(1)| \in \mathbb{Q}$ is an invariant of the surface $S$. Note that $KJ$ is defined only up to absolute value, reflecting the sign discrepancy observed by Jacobsson.

Since $\Phi_S$ is a graded map of $q$-degree $\chi(S)$, $KJ(S) = 0$ for all $S$ that are not tori. Furthermore, Rasmussen [8] and Tanaka [9] separately proved:

**Theorem 17** (Rasmussen, Tanaka). Let $S$ be a surface in $B^4$. Then $KJ(S) = 2$ if $g(S) = 1$ and $KJ(S) = 0$ otherwise.

This ended all debate about $KJ$.

### 7.2. Surface Knots with Boundary

In a slightly more general application of the Khovanov-Jacobsson technology, consider the homomorphism $\overline{\Phi}_S : \mathbb{Q} \to Kh(\partial S)$ induced by a spanning surface $S \subseteq S^3 \times I$, ...
with boundary \( L = \partial S \subseteq S^3 \times \{1\} \). Just as in the case of closed surfaces, the homology class \( KJ(S) = |\Phi_S(1)| \) gives an invariant of the boundary preserving isotopy class of the surface \( S \) in \( B^4 \).

The application of \( KJ \) to distinguishing spanning surfaces in \( B^4 \) is our primary goal for the remainder of this paper. We must first define our notion of equivalence.

In general, there are two notions of equivalence for spanning surfaces in \( B^4 \). Surfaces \( S_1 \) and \( S_2 \) spanning a link \( L \) are weakly equivalent if there exists an isotopy of \( B^4 \) taking \( S_1 \) to \( S_2 \). The surfaces \( S_1 \) and \( S_2 \) are strongly equivalent if there exists such an isotopy of \( B^4 \) taking \( S_1 \) to \( S_2 \) that keeps \( L \) fixed throughout.

Since \( KJ \) is an invariant of surfaces in \( B^4 \) up to boundary preserving isotopy, it offers an obstruction to strong equivalence. This is to say that if \( KJ(S_1) \neq KJ(S_2) \), then \( S_1 \) and \( S_2 \) are not strongly equivalent in \( B^4 \).

In the coming sections we will explore several classes of spanning surfaces for which \( KJ \) can reasonably be computed. We will begin with a few general results that follow rather directly from the definitions and Theorem 17.

**Proposition 18.** If a knot \( K \) has a spanning surface \( S \) of genus 1 and \( KJ(S) = 0 \), then \( K \) is not slice.
Proof: If $K$ were slice, then capping $K$ off outside $B^4$ with a slice $B$ and inside $B^4$ with $S$ would result in a closed surface $B^{-1} \circ S$ of genus 1. We have $KJ(B^{-1} \circ S) = 2$ by Theorem 17, but we also know that $\overline{\Phi_{B^{-1} \circ S}}(1) = \overline{\Phi_B} \circ \overline{\Phi_S}(1)$ and $\overline{\Phi_S}(1) = 0$. \qed

In a similar vein, we have

**Proposition 19.** If $S$ is a slice on a knot $K$, then $KJ(S) \neq 0$.

*Proof:* This time cap $K$ off inside $B^4$ with $S$ and outside $B^4$ with a surface $S'$ formed by gluing together a punctured $S$ with a punctured standard torus $T^*$ along their unknotted boundaries. The resulting surface $S' \circ S$ is closed with genus 1, so $\overline{\Phi_{S' \circ S}}(1) = \overline{\Phi_{S'}}(\overline{\Phi_S}(1)) = 2$ according to Tanaka/Rasmussen. Thus $\overline{\Phi_S}(1) \neq 0$. \qed

We now turn to the simplest class of links available for analysis.

**Theorem 20.** Suppose that $S \subset B^4$ is a connected surface with boundary $U^n \subset S^3$, an unlink of $n$ components in the standard crossingless diagram. Then $\Phi_S(1)$ is completely determined by the genus of $S$. In particular,

- If $g(S) = 0$, then $\Phi_S(1)$ is a $pqr$-chain on the $n$ circles in $D$.
- If $g(S) = 1$, then $\Phi_S(1)$ is twice the all $x$ state.
- If $g(S) \geq 2$ then $\Phi_S(1) = 0$. 
Note that in the chain complex for a crossingless diagram such as \( U^n \), any nontrivial state will represent a non-trivial homology class, so the first two cases above are examples of surfaces with non-trivial \( KJ \) classes.

**Proof:** Recall that \( \Phi_S \) shifts \( q \) by \( \chi S \). This means that

\[
q\Phi_S(1) = 0 + \chi S = 2 - n - 2g(S).
\]

If \( g(S) = 0 \) we get \( q\Phi_S(1) = 2 - n \). Choose an ordering on the components of \( U_n \). The simple states on \( U_n \) at this \( q \)-grading are precisely those states \( t_i \) in which the \( i \)th circle is labeled 1 and all others are labeled \( x \). Thus

\[
\Phi_S(1) = \sum_{i=1}^{n} a_i t_i
\]

for some rational coefficients \( a_i \). To determine the values of the \( a_i \) we will again consider capping \( S \) off outside \( B^4 \) in various ways.

First let \( C_k \) be an \( n \) component surface with \( \partial C_k = U^n \), the components of which are all standard caps except for the surface on the \( k^{th} \) boundary component, which is a punctured torus. Assume that \( C_k \) is positioned in time so that the caps all occur below the first saddle in the punctured torus.
Note that $\Phi_{C_k} t_i = 0$ whenever $i \neq k$, since $\Phi_{C_k}$ applies an $i$ to the $i$th component, which in $t_i$ is labeled with a 1. On $t_k$ the surface $C_k$ caps off all the circles labeled $x$ and then glues a punctured torus to the circle labeled 1, resulting in $\Phi_{C_k}(t_k) = 2$. Thus

$$\pm 2 = \Phi_{C_k \circ S}(1) = \Phi_{C_k} \sum_{i=1}^{n} a_i t_i = \Phi_{C_k} a_k t_k = 2a_k.$$ 

From this we conclude that each $a_i = \pm 1$.

To see that the $a_i$ must all have the same sign, consider capping $S$ off instead with the surface $C_{j,k}$, which is an $n-1$ component surface consisting of caps on all but the $j^{th}$ and $k^{th}$ circles, which are merged with a saddle. Again, $\Phi_{C_{j,k}} t_i = 0$ whenever $i \neq j,k$ for the same reason as before. On both $t_j$ and $t_k$ the saddle in $C_{j,k}$ merges circles $i$ and $j$ before capping off the resulting circle. Thus $\Phi_{C_{j,k}} t_j = 1 = \Phi_{C_{j,k}} t_k$, and

$$\pm 2 = \Phi_{C_{j,k} \circ S}(1) = \Phi_{C_{j,k}} \sum_{i=1}^{n} a_i t_i = \Phi_{C_{j,k}} a_j t_j + \Phi_{C_{j,k}} a_k t_k = a_j + a_k.$$ 

Thus all $a_k$ are of the same sign, and $\Phi_S(1)$ is a $pqr$-chain on $U^n$.

If instead $g(S) = 1$, $q \Phi_S(1) = 2 - n - 2g(S) = -n$. The only labeling on $U^n$ in this $q$ grading is the all $x$ state $t_x$. Suppose that $\Phi_S(1) = bt_x$. Capping $S$ off with a surface $C$ that is a union of $n$ simple caps results again in a torus, so $\Phi_{C \circ S}(1) = \pm 2$. But $\Phi_C(t_x) = 1$. Thus $b = \pm 2$. 
In the case that $g(S) \geq 2$, $q\Phi_S(1) < -n$. But $CKh^{0,l}(D) = 0$ when $l < -n$, so $\Phi_S(1) = 0$. □

**Corollary 21.** Suppose $S$ is a knotted disk in $B^4$ with $\partial S = S^1 \subset S^3$. Then $KJ(S) = t_1$ (the state given by the single circle in $\partial S$ labeled 1).

This is just a special case of Theorem 20, where $S$ has genus 0 and 1 boundary component. Note that a one element $pqr$-chain is a single circle labeled 1. It is worth mentioning that this result is consistent with Proposition 19.

This allows us the following result.

**Theorem 22.** The invariant $KJ$ cannot detect local knottedness.

**Proof:** Let $S$ be a locally knotted spanning surface for a link $L$. Decompose $S = S' \circ D_k$ where $D_k$ is a knotted disk in $S$. Corollary 21 tells us that $\Phi_{D_k}(1) = t_1$, the simple state given by the single circle labelled 1. Thus we find that $KJ(S' \circ D_k) = \Phi_{S'}(\Phi_{D_k}(1)) = \Phi_{S'}(t_1)$.

If, however, we cut $D_k$ out along its unknotted boundary and replace it with a standard disk $D$, we also get that $KJ(S' \circ D) = \Phi_{S'}(\Phi_D)(1) = \Phi_{S'}(t_1)$. Hence $KJ$ does not detect the knotting in $D_k$. □
This tells us that if $KJ$ is going to distinguish surface knots in $B^4$, an example in which the knotting is not local will be required.

7.3. **Two Slices for $9_{46}$**. We now turn to our first more complicated example. Consider the diagram $D$ for the knot $9_{46}$ pictured. This knot has two slices corresponding to the two arcs shown: a saddle at each arc separates the diagram into two clearly unlinked components. The slices $S_l$ and $S_r$ corresponding to these arcs can be described by the movie clips shown in Figure 27. We claim that $S_l$ and $S_r$ are strongly inequivalent in $B^4$. Notice that both slices contain two self-intersection arcs. These cannot be avoided: indeed, the knot $9_{46}$ has genus 1 in $\mathbb{R}^3$.

![Movie sequences for two slices $S_r$ and $S_l$ of $9_{46}$](image)

**Figure 27.** Movie sequences for two slices $S_r$ and $S_l$ of $9_{46}$.

Since $S_l$ and $S_r$ are both slices, Proposition 19 tells us that $KJ(S_l)$ and $KJ(S_r)$ are each non-trivial. To prove that $KJ(S_l) \neq KJ(S_r)$, we will first have to explicitly calculate a representative for each. The full
calculations can be found in Appendix B. The resulting representatives are given below.

\[
\Phi_{S_r}(1) = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image1.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image2.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image3.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image4.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image5.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image6.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image7.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image8.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image9.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image10.png}
\end{array}
\end{array} \end{array}
\]

\[
\Phi_{S_l}(1) = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image11.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image12.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image13.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image14.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image15.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image16.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image17.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image18.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image19.png}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{image20.png}
\end{array}
\end{array} \end{array}
\]

A word of caution: the two classes shown actually exist in different (chain homotopy equivalent) complexes, since the order in which the crossings were created in the move clips differ. In order to compare them, we must first map \( \Phi_{S_l}(1) \) into the chain complex for \( 9_{46} \) with the crossing order corresponding to the movie sequence for \( S_r \), via a homotopy equivalence such as those described in Section 3.4. This chain homotopy equivalence will change the signs on select summands in \( \Phi_{S_l}(1) \), the specifics of which will not matter in the argument to come.

To show that \( \Phi_{S_r}(1) - \Phi_{S_l}(1) \) represents a non-trivial class in \( Kh(9_{46}) \), map this difference to a simpler knot \( K \), using a chain map \( \tau \) composed of consecutive positive trims of the three crossings in the left-most column in \( D \). The map \( \tau \) and its effects are shown on page 110.
The effect of $\tau$ on $\Phi_{S_r}(1) - \Phi_{S_l}(1)$ is to annihilate all but a single summand. This surviving simple state is non-degenerate, purely negative, and labeled 1 on all vertices, so Proposition 10 says it represents a non-trivial class in $Kh(K)$. Thus $\Phi_{S_r}(1) - \Phi_{S_l}(1)$ must also represent a non-trivial class in $Kh(9_{46})$, meaning that $KJ(S_l) \neq KJ(S_r)$ and the two slices on $9_{46}$ are not strongly equivalent in $B^4$. 
8. Equivalence of Spanning Surfaces in $S^3$

All of the results involving $KJ$ classes that have been presented thus far have been for surfaces in $B^4$ with boundary in $S^3$. We will now focus on surfaces that are embedded in $S^3$.

Just as in the $B^4$ case, there are two distinct notions of equivalence for spanning surfaces in $S^3$. Surfaces $S_1$ and $S_2$ spanning a link $L$ are **weakly equivalent in $S^3$** if there exists an isotopy of $S^3$ taking $S_1$ to $S_2$ and **strongly equivalent in $S^3$** if there exists such an isotopy that is boundary preserving.

When surfaces embedded in $S^3$ are pushed into $B^4$, they have a tendency to simplify. In particular, strongly inequivalent surfaces of the same genus in $S^3$ do not generally remain strongly inequivalent when pushed into $B^4$. In fact, there were previously no known examples of homeomorphic surfaces $S_1$ and $S_2$ that are strongly inequivalent in $S^3$ and that remain strongly inequivalent when pushed into $B^4$. We will provide a family of such examples in Section 8.2.

First, however, we will analyze a class of simpler surfaces in $S^3$ for which we can calculate $\Phi_S(1)$.

8.1. **Seifert Surfaces.** Consider a diagram $D$ for an oriented link $L$. Generate a Seifert surface $S_D$ for $L$ using Seifert’s algorithm: smooth $D$ according to its orientation and consider the resulting Seifert circles
to be the boundaries of embedded disks (with the convention that the
disks are stacked if the circles are nested, with the outer circle posi-
tioned beneath). Build $S_D$ from these disks by attaching half twisted
bands between them in the location and direction dictated by the cross-
ings in $D$. A surface that is the result of such a construction for some
diagram will be called a Seifert surface for the diagram $D$.

When calculating $KJ(S_D)$, build $S_D$ up from basic cobordisms in
stages, according to the following prescription:

(1) First, birth the $n$ Seifert circles, in order by nesting depth, with
the outer circles birthed first.

(2) For each negative crossing $c$ in $D$, add an appropriately oriented
Reidemeister 1 twist to one of the two Seifert circles adjacent to
c. Finish re-creating $c$ by merging (with a saddle) this new twist
with the opposite Seifert circle. This combined twist-and-glue
move and the map that it induces on generators are pictured in
Figure 28.

(3) Recreate the positive crossings with positive twist-and-glue moves,
as shown in Figure 29.

This recipe allows us to compute a representative for $KJ(S_D)$ using
only births and $R1$ twist-and-glue moves.

A few results regarding $\Phi_{S_D}(1)$ are immediate.
Observation 23. The representative $\Phi_{SD}(1)$ is a sum of simple states, each associated to the unique orientation preserving smoothing $M_D$ of $D$.

The homomorphisms induced by the twist-and-glue moves act as the identity on the underlying manifolds; they merely change the labels in some cases. There might be a bifurcation of terms, but each summand will be a labeling of $M_D$.

Observation 24. The height $h(\Phi_{SD}(1)) = 0$. 

Observe that in $M_D$, the negative crossings are precisely those that get resolved as 1-smoothings. Thus $h(\Phi_{S_D}(1)) = n_--n_- = 0$.

The crossing information in $\Gamma_D$ directly determines the labeling on $\Phi_{S_D}(1)$. Following Seifert’s algorithm, decompose $S_D = S_3 \circ S_2 \circ S_1$, where $S_1$ is the birth of the Seifert circles, $S_2$ consists of the composition of all negative twist-and-glue moves, and $S_3$ is the composition of the positive twist-and-glue moves. The image $\Phi_{S_2 \circ S_1}(1)$ is the oriented smoothing $M_D$ with each circle labelled with a 1. The order in which the positive twist-and-glue moves are positioned in time may effect the resulting representative $\Phi_{S_3}(\Phi_{S_2 \circ S_1}(1))$, but the invariance of $KJ$ under boundary preserving isotopy guarantees our choice of order will not effect $KJ(S_D)$. This gives us the leeway to choose an ordering for the position of these positive twist-and-glue moves in time, allowing for:

**Theorem 25.** The labels on the representative $\Phi_{S_D}(1)$ are completely determined by $\Gamma_{\partial M_D}^+$. In particular, for a connected $\Gamma_{\partial M_D}^+$,

- If $B^1(\Gamma_{\partial M_D}^+) = 0$, then $\Phi_{S_D}(1)$ is given by a $pQr$-chain on $\Gamma_{M_D}^+$ and 1’s on all non-positive vertices. The terms in the $pQr$-chain have signs determined by distance within $\Gamma_{M_D}^+$ from a fixed leaf $v_0$. 
• If $B^1(\Gamma^+_M) = 1$, then $\Phi_{S_D}(1) = 2t_x$, where $t_x$ is $M_D$ with all positive vertices labeled $x$ and all non-positive vertices labeled $1$.

• If $B^1(\Gamma^+_M) \geq 2$, $\Phi_{S_D}(1) = 0$.

Proof: We need only determine $\Phi_{S_3}$. Begin with the case where $B^1(\Gamma^+_M) = 0$. Choose an ordering on $E^+_M = (e_1, e_2, \ldots, e_k)$ such that $e_1$ is the edge adjacent to the leaf $v_0$ of $\Gamma^+_M$, and the subsequent edges are partially ordered by distance from $e_1$ in $\Gamma^+_M$.

Position $S_3$ so that it decomposes as $S_3 = S_{e_k} \circ \cdots \circ S_{e_2} \circ S_{e_1}$, where $S_{e_i}$ is the positive twist-and-glue cobordism at the $i^{th}$ edge. The image of $\Phi_{S_{e_1}} \circ \Phi_{S_{e_2}}(1)$ is the difference of two states as shown in Figure 30.

Once a circle is labeled with an $x$, a positive twist-and-glue move made away from that circle will force an $x$ onto the subsequent circle, while the term labeled 1 will again be split as in the first step, introducing another negative term.

Proceed recursively until all of $S$ is re-created. The result of building out such a tree is a $pQr$-chain, in which the terms with negative coefficients are precisely those whose corresponding circles are of odd distance from $v_0$ in $\Gamma^+_M$.

If instead $B^1(\Gamma^+_M) \geq 1$, choose a maximal tree $G$ in $\Gamma^+_M$. Isotope $S$ so that its early stages are induced by the tree $G$, resulting in a $pQr$-chain as in the first case described above. Since $G$ spans $\Gamma^+_M$ and
If there happens to exist a second edge $e' \in \Gamma^+_D$ not contained in $G \cup e$, this edge must connect two circles that are already labeled $x$ in both surviving summands in $\Phi S_{Gue}$. The twist-and-glue move $\Phi S_e$ will then annihilate both terms, leaving $\Phi S_{D}(1) = 0$. $\square$
Seifert surfaces offer many examples with non-trivial $KJ$. For instance, any purely negative or purely positive diagram $D$ will have non-trivial $KJ(S_D)$. The non-trivial state $t$ presented in Figure 26 on page 71 represents the Seifert class for the standard diagram for the 6$_2$ knot.

We also get a pair of slice obstructions for pretzel knots. The first is a slightly weaker version of a theorem proven in [4] using Heegaard-Floer invariants.

**Corollary 26.** Let $p, q, r$ be odd integers that are either all positive or all negative. Then the $(p, q, r)$-pretzel knot $K$ is not slice.

**Proof:** Suppose $p, q, r < 0$. Let $D$ be the usual projection of $K$ and $S_D$ the usual genus 1 surface spanning $K$. All edges in $M_D$ are positive, so $B^1(\Gamma^+_{MD}) = 2$ and hence $KJ(S_D) = 0$ by Theorem 25. Thus $K$ is not slice according to Proposition 18. If instead $p, q, r > 0$, note that the mirror of $K$ is the $(-p, -q, -r)$ pretzel, which is not slice. Thus $K$ is not slice either. \qed

**Corollary 27.** Let $p, q \leq -3$ be odd integers. The $(p, q, 1)$-pretzel knot is not slice. Also, the $(-p, -q, -1)$-pretzel is not slice.
Proof: Again, take $D$ to be the usual projection of $K$ and $S_D$ the genus 1 Seifert surface for $K$. The $p + q$ left handed twists correspond to a loop in $\Gamma_M^+$, meaning that $\Phi_{S_D}(1) = 2t_x$. The single right handed twist corresponds to a negative edge that is adjacent to two positive vertices, each labelled $x$ in $\Phi_{S_D}(1)$. Since $p, q \leq -3$, these positive vertices are not adjacent in $\Gamma_M^+$, so Proposition 12 implies that $KJ(S_D) = 0$. □
8.2. **A Link With Distinct Spanning Surfaces of Genus 2.** We will now present an oriented link that admits two spanning surfaces in $S^3$ with distinct $KJ$ classes.

The link $L$ shown has two spanning surfaces of genus 2 that are illustrated in Figure 31. The surface $S_1$ at the left is evident from the diagram. The surface $S_2$ is a bit more complicated to illustrate. It can be created from $S_1$ by removing the two twisted bands in the clasp at the left and replacing them with a sleeve that encompasses the band region, as shown. For a more thorough description of this sleeve, see Figure 35, in which the sleeve is decomposed as a link cobordism movie.

**Figure 31.** The surfaces $S_1$ and $S_2$ that span $L$. 
Theorem 28. The two surfaces $S_1$ and $S_2$ spanning $L$ in $S^3$ have $KJ(S_1) \neq KJ(S_2)$, meaning that not only are $S_1$ and $S_2$ not strongly equivalent in $S^3$ but they remain inequivalent even when pushed into $B^4$.

To prove this, we will calculate a representative for the $KJ$ class of each surface and compare these representatives by applying the techniques presented in Section 6.4.

8.2.1. Computing $KJ(S_1)$ and $KJ(S_2)$. We will begin by carefully choosing a movie sequence that describes each $S_i$.

The relatively large crossing number of $D$ makes the calculation of $\Phi_{S_i}(1)$ potentially rather painful. To minimize the complexity of this calculation we will focus on five special tangles within $D$. The tangle in the middle left will be referred to as the clasp tangle. The two tangles to the middle right will be referred to as star tangles. The crossingless tangles at the top and bottom will be referred to as arm tangles. All other crossings in $D$ will be referred to as buffer crossings.

We will now describe a movie sequence for $S_1$. The idea is to generate the portions of $S_1$ that are within each of the insulated tangles
separately and then glue them together with negative twist-and-glue moves at the buffer crossings.

Begin with the crossed ribbons seen within the star tangles. Each such piece of surface can be created with what we will call a star move, which decomposes as the births of a pair of non-concentric disks followed by two Reidemeister 2 tucks that slide one disk under the other. The star move and its induced map $\Phi_{st}$ are shown in Figure 32.

The clasp in $S_1$ can be described in a similar fashion, with a birth followed by two twisted bands and a saddle. This clasping move is illustrated, with its induced map $\Phi_{cl}$, in Figure 33.
Figure 33. Generating the clasp in $S_1$.

We can use $D$ to identify the disjoint balls in $\mathbb{R}^2$ in which we will perform the star moves and the clasp, allowing us to describe $S_1$ with the following sequence of movie clips:

1. From the empty link, birth and twist the clasp surface.
2. Follow this with a pair of subsequent star moves, one within each of the star regions.
3. Birth circles in each of the arm tangles.
4. After a slight isotopy to bring the various discs together at the buffer crossings, connect the five disjoint regions of surface with a sequence of negative twist-and-glue moves at each of the negative buffer crossings.
This describes a frame by frame link cobordism movie for the spanning surface $S_1$. We can therefore calculate $\Phi_{S_1}(1)$ using the maps induced by these clips, extending each to be the necessary identity outside of its own tangle.

\[ \Phi_{S_1}(1) \text{ for } KJ(S_1). \]

The fact that a negative twist-and-glue move creates a negative edge without changing the labels on the incident circles (see page 85) tells us that $\Phi_{S_1}(1)$ is given by the tangle states illustrated above attached to one another with negative edges. The full representative for $KJ(S_1)$ is given in Figure 34, in which the star tangles are labelled with $\Phi_{st}$ to indicate that each actually represents the sum of five different tangle states produced by the corresponding star move. Thus $\Phi_{S_1}(1)$ is actually the sum of all 100 simple states formed by joining one of the
five summands within each star with one of the four terms in the clasp $pQr$-chain, attached by the exterior tangle state shown.

Figure 35. A movie sequence for the sleeve clasp in $S_2$. 
At first glance, $S_2$ appears much harder to describe with a movie. It turns out, however, that the sleeve does not complicate the calculation as much as it might seem. The key is to generate the sleeve before generating the other portions of $S_2$. This can be done with the sequence of moves shown in Figure 35.

The first three of these frames can be thought of as being carried out completely within the region where the sleeve will be. Start by birthing a disk within the future sleeve, positioned near the top of the clasp. A saddle then splits this circle into two, creating an annulus whose boundary components are the two circles shown. One of these boundary components approaches the clasp immediately, while the other widens and moves around the star regions, carving out the rest of the sleeve before narrowing and approaching the clasp region from below.

These two circles then pass through each other in the fashion shown to create the rest of the clasp. This combined birth, split, and clasp sequence describes the beginning of a movie sequence for $S_2$. The rest of the sequence mimics the movie for $S_1$ as if the sleeve were not there. We decompose $S_2$ as follows:
(1) Birth, split, and clasp, as just described.

(2) Perform a star move at each of the star tangles in $D$, each in the same relative orientation as the corresponding star move used to generate $S_1$.

(3) Birth circles in each of the arm tangles.

(4) After a slight isotopy to bring the various discs together at the buffer crossings, connect the five disjoint regions of surface with a sequence of negative twist-and-glue moves at each of the negative buffer crossings.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure36}
\caption{A representative $\Phi_{S_2}(1)$ for $KJ(S_2)$.}
\end{figure}

With this movie for $S_2$ we again build $\Phi_{S_2}(1)$ from its composite parts as previously described, connected with negative edges. This time the star tangles each contribute five tangle states and the clasp tangle contributes 16 tangle states, so $\Phi_{S_2}(1)$ is the sum of 400 simple
states formed by replacing each of the tangles in Figure 36 with one summand from $\Phi_{st}(1)$ or $\Phi_{c2}(1)$, as appropriate.

8.2.2. Distinguishing $KJ(S_1)$ from $KJ(S_2)$. Just as we did in Section 7.3 in the $9_{46}$ example, we will concoct a sequence of trimming maps $\tau$ that will annihilate $\Phi_{S_1}(1)$ while mapping $\Phi_{S_2}(2)$ to a class that is known to be non-trivial. These $\tau$ maps will consist of selective positive trimming (see page 69) maps defined within each of the star and clasp tangles in $D$.

We can again ignore the complications that arise from varying the crossing order, as explained on page 82, since our trimming map will annihilate each of the individual generators for $\Phi_{S_1}(1)$.

![Diagram](image)

**Figure 37.** Trim a pair of crossings within each star tangle.
Consider first the map $\tau_{st}$ given by a composition of two positive edge trims within a star tangle. The simple state circled in Figure 37

**Figure 38.** Trim three crossings within the clasp tangle.
is the one that we wish to survive the trimming, so the crossings to be trimmed should be chosen relative to this particular smoothing, as shown.

Now define a chain map $\tau_c$ that is defined within the clasp tangle by a sequence of three consecutive positive crossing trims. This movie clip and the map it induces on $CKh$ are presented in Figure 38. Note that this is the same chain map that was used to distinguish the two slices of $9_{46}$.

![Diagram](image)

The composition $\tau_c\tau_{st_1}\tau_{st_2}$ annihilates $\Phi_{S_1}(1)$ and takes $\Phi_{S_2}(1)$ to the simple state shown above. This simple state is non-degenerate, even, connected, purely negative, and has exactly one circle labeled $x$. Thus Theorem 8 tells us that $\tau_c\tau_{st_1}\tau_{st_2}\Phi_{S_2}(1)$ is non-trivial. We therefore conclude that $KJ(S_1)$ and $KJ(S_2)$ are distinct classes, and $S_1$ and $S_2$ are not strongly equivalent even when pushed into $B^4$. 
8.2.3. A family of examples. This example can be easily generalized to a family of more complicated two-component links, each with multiple distinct spanning surfaces of genus 2. Consider the oriented tangle diagram shown in Figure 39.

\[ L(K_1, K_2, n) \]

Let \( L(K_1, K_2, n) \) be a link described by this diagram, in which the three empty rectangles are filled in as follows: The middle tangle contains a chain of \( 2n \) star tangles, with negative buffer crossings between them. The diagram excerpt shows four stars. The top and bottom tangles contain buffered doubles of the knots \( K \) and \( K' \), respectively. A buffered double is gotten from a blackboard double by inserting a negative half-twist on each pair of parallel strands connecting the stars within the double. The buffered double of a trefoil is shown.

**Figure 39.** A family of links.
Each such $L(K_1, K_2, n)$ will be spanned by surfaces analogous to $S_1$ and $S_2$ that have distinct Khovanov-Jacobsson classes. The argument in each case is almost identical to the proof presented above; there will be more star tangles in $D$, so more star moves in the movie clips generating $S_1$ and $S_3$ and more star trims in the representation map. In any case, the post-representation state will be a simple state that meets the criteria presented in Theorem 8 for non-triviality.
Figure 40. The chain homotopy equivalence induced by the negative Reidemeister twist/untwist.
Figure 41. The chain map induced by the positive Reidemeister twist/untwist.
Figure 42. The chain map induced by the Reidemeister tuck/untuck.
Figure 43. The map induced by one version of the third Reidemeister move.
Figure 44. The map induced by the other version of
the third Reidemeister move.
Figure 45. $CKh(D)$ and $Kh(L)$ for the Hopf link.
Figure 46. Knot $9_{46}$: $\Phi_{S_i}$ in all its glory
Figure 47. Knot 9_{46}: $\Phi_S$, in all its glory.
A movie sequence for a cobordism $C$ and the map it induces on $KJ(S_r) - KJ(S_t)$. 

\[
\begin{align*}
&\text{A movie sequence for a cobordism } \\
&C \text{ and the map it induces on } \\
&KJ(S_r) - KJ(S_t).
\end{align*}
\]
References


