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Jan Stolarek
Simon Peyton Jones
Richard A. Eisenberg
Bryn Mawr College, rae@cs.brynmawr.edu

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Injective Type Families for Haskell (extended version)

Jan Stolarek
Politechnika Łódzka
jan.stolarek@p.lodz.pl

Simon Peyton Jones
Microsoft Research Cambridge
simonpj@microsoft.com

Richard A. Eisenberg
University of Pennsylvania
eir@cis.upenn.edu

Abstract

Haskell, as implemented by the Glasgow Haskell Compiler (GHC), allows expressive type-level programming. The most popular type-level programming extension is *TypeFamilies*, which allows users to write functions on types. Yet, using type functions can cripple type inference in certain situations. In particular, lack of injectivity in type functions means that GHC can never infer an instantiation of a type variable appearing only under type functions.

In this paper, we describe a small modification to GHC that allows type functions to be annotated as injective. GHC naturally must check validity of the injectivity annotations. The algorithm to do so is surprisingly subtle. We prove soundness for a simplification of our algorithm, and state and prove a completeness property, though the algorithm is not fully complete.

As much of our reasoning surrounds functions defined by a simple pattern-matching structure, we believe our results extend beyond just Haskell. We have implemented our solution on a branch of GHC and plan to make it available to regular users with the next stable release of the compiler.

Categories and Subject Descriptors F.3.3 [Logics And Meanings Of Programs]: Studies of Program Constructs – Type structure; D.3.1 [Programming Languages]: Formal Definitions and Theory – Semantics; D.3.2 [Programming Languages]: Language Classifications – Haskell

Keywords Haskell; type-level programming; type families; functional dependencies; injectivity

1. Introduction

The Glasgow Haskell Compiler (GHC) offers many language extensions that facilitate type-level programming. These extensions include generalized algebraic data types (GADTs) (Cheney and Hinze 2003; Peyton Jones et al. 2006), datatype promotion with kind polymorphism (Yorgey et al. 2012), and functional dependencies (Jones 2000). But the most widespread\(^1\) extension for type-level programming is for *type families*, which allow users to define type-level functions (Chakravarty et al. 2005a,b; Eisenberg et al. 2014) run by the type checker during compilation. Combined with other features, they allow expressiveness comparable to that of languages with dependent types (Lindley and McBride 2013).

However, type families as implemented in GHC have a serious deficiency: they cannot be declared to be *injective*. Injectivity is very important for type inference: without injectivity, some useful functions become effectively unusable, or unbearably clumsy. *Functional dependencies*, which have been part of GHC for many years, are arguably less convenient (Section 7), but they certainly can be used to express injectivity. That leaves programmers with an awkward choice between the two features.

In this paper we bridge the gap, by allowing programmers to declare their type functions injective, while the compiler checks that their claims are sound. Although this seems straightforward, it turned out to be much more subtle than we expected. Our main contribution is to identify and solve these subtleties. Although our concrete focus is on Haskell, our findings apply to any language that defines functions via pattern matching and allows to run them during compilation. Specifically:

- We introduce a backwards-compatible extension to type families, which allows users to annotate their type family declarations with information about injectivity (Section 3).
- We give a series of examples that illustrate the subtleties of checking injectivity (Section 4.1).
- We present a compositional algorithm for checking whether a given type family (which may be open or closed) is injective (Section 4.2), and prove it sound (Section 4.3). We show that a compositional algorithm cannot be complete, but nevertheless give a completeness proof for a sub-case where it holds (Section 4.4).
- We explain how injectivity information can be exploited by the type inference algorithm, including elaboration into GHC’s statically typed intermediate language, System FC (Section 5).
- We describe how to make the injectivity framework work in the presence of kind polymorphism (Section 6).
- We provide an implementation of our solution in a branch of GHC. We expect it to become available to regular users with the next stable release.

Our work is particularly closely related to functional dependencies, as we discuss in Section 7, leaving other related work for Section 8. An extended version of the paper is available online, with proofs of the theorems (Stolarek et al. 2015).

2. Why injective type families are needed

We begin with a brief introduction to type families, followed by a motivating example, inspired by a real bug report, that illustrates why injectivity is important.

2.1 Type families in Haskell

Haskell (or, more precisely, GHC), supports two kinds of type family: *open* and *closed*.\(^2\) An open type family (Chakravarty et al. 2005a,b) is specified by a *type family* declaration that gives its arity, its kind (optionally), and zero or more *type instance* declarations that give its behaviour. For example:

\[ \text{associated types} \text{ (Chakravarty et al. 2005a) are essentially syntactic sugar for open type families. Everything we say in this paper works equally for associated types, both in theory and in the implementation. So we do not mention associated types further, apart from a short discussion in Section 3.3.} \]

\(^1\) This is an extended version of the submitted work.

\(^2\) Appendix A gives data and describes our methodology for obtaining them.
The type-instance equations may be scattered across different modules and are unordered; if they overlap they must be compatible. We say that two overlapping type family equations are compatible when any application matching both of these equations reduces, in one step, to the same result with any of these equations.

A closed type family (Eisenberg et al. 2014) is declared with all its equations in one place. The equations may overlap, and are matched top-to-bottom. For example:

\[
\text{type family } F \ a \\
\text{type instance } F \ \text{Int} = \text{Bool} \\
\text{type instance } F \ [a] = a \rightarrow a
\]

A practical use case for injectivity arises in the vector library. In the real library, the argument to \texttt{Mutable} is the type constructor for a vector; but element polymorphism complicates the example

\[
\sigma, \tau ::= \alpha \quad \text{Type variable} \\
H \quad \text{Type constructor} \\
\tau_1 \tau_2 \quad \text{Application} \\
F \tau \quad \text{Saturated type-function application}
\]

\[\text{Figure 1. Syntax of types.}\]

\[
\text{freeze } :: \text{Mutable } v \rightarrow \text{IO } v \\
\text{convert } :: (\text{Vector } v, \text{Vector } w) \Rightarrow v \rightarrow w
\]

\[\text{freeze takes a mutable vector and turns it into an immutable one; convert converts one kind of vector into another. But now suppose the programmer writes this:}\]

\[
\text{ftc} :: (\text{Vector } v, \text{Vector } w) \Rightarrow \text{Mutable } v \rightarrow \text{IO } w \\
\text{ftc } \text{mv} = \text{do } [v \leftarrow \text{freeze } \text{mv}; \text{return } (\text{convert } v)]
\]

Currently GHC complains that the type of \texttt{ftc} is ambiguous. Why? Consider a call (\texttt{ftc vec}), where \texttt{vec :: MByteString}. What type should instantiate \texttt{v} in the call? Since \texttt{MByteString = MByteString}, picking \texttt{v \rightarrow ByteString} would certainly work. But if \texttt{Mutable} were not injective, there could be other valid choices for \texttt{v}, say \texttt{v \rightarrow ByteStringX}. But different choices would give different behaviour, because the class instances for \texttt{Vector ByteString} and \texttt{Vector ByteStringX} might behave quite differently. Since there is no unique choice, GHC refrains from choosing, and instead reports \texttt{v} as an ambiguous type variable.

To resolve the ambiguity of \texttt{v} we have to give guidance to the type inference engine. A standard idiom in cases like these is to use proxy arguments. We could rewrite \texttt{ftc} like this:

\[
\text{data Proxy } a \\
\text{ftc'} :: (\text{Vector } v, \text{Vector } w) \Rightarrow \text{Proxy } v \rightarrow \text{Proxy } w \\
\text{ftc'} \text{ mv} = \text{do } [v \leftarrow \text{freeze } v; \text{return } (\text{convert } v)]
\]

The value of the proxy argument is \(\bot\); its only role is to tell the type inference engine to instantiate the type variable \(v\) to \textit{ByteString}.

This works, but it is absurdly clumsy, forcing the programmer to supply redundant arguments whose value is (in the programmer’s mind) unambiguously determined by the context.

_Like this:_

\[
\text{ftc} = \text{Proxy } v \\
\text{ftc'} = \text{Proxy } v
\]

The types \(\tau\) and \(\sigma\) must be monotypes; they contain no for-all quantifiers.

In addition, the types \(\tau\) and \(\sigma\) on the left hand side must be type-function-free.

The left hand side (LHS) of the equation is \(F \ \tau\) and the right hand side (RHS) is \(\sigma\).

For the purposes of Sections 3–5 we restrict our attention to kind-monomorphic type functions. The generalization to polymorphic kinds is straightforward – see Section 6.

Type functions may only appear \emph{saturated} in types. That is, if \(F\) has arity 2, it must always appear applied to two arguments. Figure 1 gives the syntax of (mono-)types.

2.2 The need for injectivity

A practical use case for injectivity arises in the vector library. That library defines an open type family, \texttt{Mutable}, that assigns a mutable counterpart for every immutable vector type:

\[
\text{type family } \text{Mutable } v \\
\text{type instance } \text{Mutable } \text{ByteString} = \text{MByteString}
\]

For example, if \texttt{ByteString} is an immutable vector of bytes, and \texttt{MByteString} is the mutable variant, then we can express the connection by writing:

\[
\text{type instance } \text{Mutable } \text{ByteString} = \text{MByteString}
\]

In the real library, the argument to \texttt{Mutable} is the type constructor for a vector; but element polymorphism complicates the example

\[
\sigma, \tau ::= \alpha \quad \text{Type variable} \\
H \quad \text{Type constructor} \\
\tau_1 \tau_2 \quad \text{Application} \\
F \tau \quad \text{Saturated type-function application}
\]

and thereby obviate the main point, so we use a monomorphic version here.

The library also provides two functions over vectors:

\[
\text{freeze } :: \text{Mutable } v \rightarrow \text{IO } v \\
\text{convert } :: (\text{Vector } v, \text{Vector } w) \Rightarrow v \rightarrow w
\]

\texttt{freeze} takes a mutable vector and turns it into an immutable one; \texttt{convert} converts one kind of vector into another. But now suppose the programmer writes this:

\[
\text{ftc} :: (\text{Vector } v, \text{Vector } w) \Rightarrow \text{Mutable } v \rightarrow \text{IO } w \\
\text{ftc } \text{mv} = \text{do } [v \leftarrow \text{freeze } \text{mv}; \text{return } (\text{convert } v)]
\]

Currently GHC complains that the type of \texttt{ftc} is ambiguous. Why? Consider a call (\texttt{ftc vec}), where \texttt{vec :: MByteString}. What type should instantiate \texttt{v} in the call? Since \texttt{MByteString = MByteString}, picking \texttt{v \rightarrow ByteString} would certainly work. But if \texttt{Mutable} were not injective, there could be other valid choices for \texttt{v}, say \texttt{v \rightarrow ByteStringX}. But different choices would give different behaviour, because the class instances for \texttt{Vector ByteString} and \texttt{Vector ByteStringX} might behave quite differently. Since there is no unique choice, GHC refrains from choosing, and instead reports \texttt{v} as an ambiguous type variable.

To resolve the ambiguity of \texttt{v} we have to give guidance to the type inference engine. A standard idiom in cases like these is to use proxy arguments. We could rewrite \texttt{ftc} like this:

\[
\text{data Proxy } a \\
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\text{ftc'} \text{ mv} = \text{do } [v \leftarrow \text{freeze } v; \text{return } (\text{convert } v)]
\]

The value of the proxy argument is \(\bot\); its only role is to tell the type inference engine to instantiate the type variable \(v\) to \textit{ByteString}.

This works, but it is absurdly clumsy, forcing the programmer to supply redundant arguments whose value is (in the programmer’s mind) unambiguously determined by the context.

Clearly, in the case of \texttt{Mutable}, the library designers intend the type family to be injective: every immutable vector type has its own unique mutable counterpart. The purpose of this paper is to allow programmers to express that intent explicitly, by providing an injectivity annotation, thus:

\[
\text{type family } \text{Mutable } v = r \mid r \rightarrow v
\]

The user names the result of \texttt{Mutable} as \(r\) and, using syntax inspired by functional dependencies, declares that the result \(r\) determines the argument \(v\). GHC then verifies that the injectivity annotation provided by the user holds for every type family instance. During type inference, GHC can exploit injectivity to resolve type ambiguity. This solves the problem in vector library in one line – no other changes are required.

3. Injective type families

Next we describe our proposed extension, from the programmer’s point of view.

3.1 Injectivity of type families

In the rest of this paper we depend on the following definition of injectivity for type families:

\textbf{Definition 1 (Injectivity).} A type family \(F\) is \(n\)-injective (i.e. injective in its \(n\)'th argument) iff \(\forall \sigma, \tau : F \ \sigma \sim F \ \tau \implies \sigma_n \sim \tau_n\).
Here $\sigma \sim \tau$ means that we have a proof of equality of types $\sigma$ and $\tau$. So the definition simply says that if we have a proof that $F\sigma$ is equal to $F\tau$, then we have a proof that $\sigma_n$ and $\tau_n$ are equal. Moreover, if we know that $F\sigma \sim F\tau'$ we can discover injective arguments $\pi_n$ by looking at the defining equations of $F$. Section 5 provides the details.

3.2 Annotating a type family with injectivity information

Injectivity is a subtle property and inferring it is not necessarily possible or desirable (see Section 3.4), so we therefore ask the user to declare it. The compiler should check that the declared injectivity of a type family is sound.

What syntax should we use for such injectivity annotations? We wanted to combine full backwards compatibility when injectivity is not used, and future extensibility (Section 7 discusses the latter). Definition 1 admits injectivity in only some of the arguments and so we have to be able to declare that a function is injective in its second argument (say) but not its first.

To achieve this, we simply allow the programmer to name the result type and, using a notation borrowed from functional dependencies (Jones 2000), say which arguments are determined by the result type and, using a notation borrowed from functional dependencies (Jones 2000), say which arguments are determined by the result type and, using a notation borrowed from functional dependencies (Jones 2000), say which arguments are determined by the result type. For example:

```
type family F a b c = r | r -> a c
```

The “$\sim r$” part names the result type, while the “$r \rightarrow a c$” termed the injectivity condition – says that the result $r$ determines arguments $a$ and $c$, but not $b$. The result variable may be annotated with a kind, of course, and the injectivity part is optional. So all of the following are legal definitions:

```
type family F a b c
  type family F a b c = r
  type family F a b c = (r :: * -> *) | r -> a
  type family F a b c = r | r -> a c
```

Examples above use open type families but the syntax also extends to closed type families, where the injectivity annotation precedes the where keyword.

3.3 Associated types

A minor syntactic collision occurs for associated types:

```
class C a b where
  type F a b
  type F a b = b
```

The second line beginning “type $F a b$” is taken as the default instance for the associated type (to be used in instances of $C$ in which $F$ is not explicitly defined). Note that the family and instance keywords can be omitted for associated types and that the default instance type $F a \equiv b$ looks suspiciously like a type family with a named result type. To avoid this ambiguity, you can only name the result type with associated types if you also give an injectivity annotation, thus:

```
class C a b where
  type F a b = \sigma | \sigma \rightarrow b
  type F a b = b
```

As explained in Section 4, GHC must check instances of injective type families to make sure they adhere to the injectivity criteria. For associated type defaults, the checks are made only with concrete instances (that is, when the default is actually used in a class instance), not when processing the default declaration. This choice of behaviour is strictly more permissive than checking defaults at the class declaration site.

3.4 Why not infer injectivity?

One can wonder why we require explicit annotations rather than inferring injectivity.

For open type families, inferring injectivity is generally impossible, as the equations are spread across modules and can be added at any time. Inferring injectivity based on only those equations in the declaring module would lead to unexpected behaviour that would arise when a programmer moves instances among modules.

Inferring injectivity on closed type families, however, is theoretically possible, but we feel it is the wrong design decision, as it could lead to unexpected behaviour during code refactoring. An injectivity declaration states that the injectivity property of a type family is required for the program to compile. If injectivity were inferred, the user might be unaware that she is relying on injectivity. Say our programmer has an inferred-injective type family $F$. She then adds a new equation to the definition of $F$ that breaks the injectivity property. She could easily be surprised that, suddenly, she has compilation errors in distant modules, if those modules (perhaps unwittingly) relied on the injectivity of $F$. Even worse, the newly-erroneous modules might be in a completely different package. With the requirement of an explicit annotation, GHC reports an error at the offending type family equation. To keep matters simple we restrict ourselves to explicitly-declared injectivity.

4. Verifying injectivity annotations

Before the compiler can exploit injectivity (Section 5), it must first check that the user’s declaration of injectivity is in fact justified. In this section we give a sound, compositional algorithm for checking injectivity, for both open and closed type functions.

We want our algorithm to be compositional or modular: that is, we can verify injectivity of function $F$ by examining only the equations for $F$, perhaps making use of the declared injectivity of other functions. In contrast a non-compositional algorithm would require a global analysis of all functions simultaneously; that is, a compositional algorithm is necessarily incomplete. A non-compositional algorithm would be able to prove more functions injective (Section 4.4), but at the expense of complexity and predictability. A contribution of this paper to articulate a compositional algorithm, and to explain exactly what limitations it causes.

Soundness means that if the algorithm declares a function injective, then it really is; this is essential (Section 4.3). Completeness would mean that if the function really is injective, then the algorithm will prove it so. Sadly, as we discuss in Section 4.4, completeness is incompatible with compositionality. Nevertheless we can prove completeness for a sub-case.

4.1 Three awkward cases

Checking injectivity is more subtle than it might appear. Here are three difficulties, presented in order of increasing obscurity.

**Awkward Case 1: injectivity is not compositional**  
First consider this example:

```
type family F1 a = r | r -> a
  type instance F1 [a] = G a
  type instance F1 (Maybe a) = H a
```

Is $F1$ injective, as claimed? Even if $G$ and $H$ are injective, there is no guarantee that $F1$ is, at least not without inspecting the definitions of $G$ and $H$. For example, suppose we have:

```
type instance G Int = Bool
  type instance H Bool = Bool
```

So both $G$ and $H$ are injective. But $F1$ clearly not injective; for example $F1 [\text{Int}] \sim G \text{Int} \sim \text{Bool} \sim H \text{Bool} \sim$
Injective Type Families for Haskell

F1 (Maybe Bool). Thus, injectivity is not a compositional property.

However, it is over-conservative to reject any type function with type functions in its right-hand side. For example, suppose G and H are injective, and consider F2 defined thus:

\[
\begin{align*}
type family & F2 \ a = r | r \to a \\
type instance & F2 \ [a] = [G \ a] \\
type instance & F2 \ (Maybe \ a) = H \ a \to Int
\end{align*}
\]

Since a list cannot possibly match a function arrow, an equality (\(F2 \ \sigma \sim F2 \ \tau\)) can only hold by using the same equation twice; and in both cases individually the RHS determines the LHS because of the injectivity of G and H. But what about these cases?

\[
\begin{align*}
type family & F3 \ a = r | r \to a \\
type instance & F3 \ [a] = Maybe \ (G \ a) \\
type instance & F3 \ (Maybe \ a) = Maybe \ (H \ a) \\
type family & F4 \ a = r | r \to a \\
type instance & F4 \ [a] = (G \ a, a, a) \\
type instance & F4 \ (Maybe \ a) = (H \ a, a, Int, Bool)
\end{align*}
\]

F3 is not injective, for the same reason as F1. But F4 is injective, because, despite calls to two different type families appearing as the first component of a tuple, the other parts of the RHS ensure that they cannot unify.

Awkward Case 2: the right hand side cannot be a bare variable or type family

The second awkward case is illustrated by this example:

\[
\begin{align*}
type family & W1 \ a = r | r \to a \\
type instance & W1 \ [a] = a
\end{align*}
\]

To a mathematician this function certainly looks injective. But, surprisingly, it does not satisfy Definition 1! Here is a counterexample. Clearly we do have a proof of (\(W1 \ [W1 \ Int] \sim W1 \ Int\)), simply by instantiating the type instance with \([a \to W \ Int]\). But if \(W1\) was injective in the sense of Definition 1, we could derive a proof of [\(W1 \ Int\)] \sim Int, and that is plainly false! Similarly:

\[
\begin{align*}
type family & W2 \ a = r | r \to a \\
type instance & W2 \ [a] = W2 \ a
\end{align*}
\]

Again \(W2\) looks injective. But we can prove \(W2 \ [Int] \sim W2 \ Int\), simply by instantiating the type instance; then by Definition 1, we could then conclude \([Int] \sim Int\), which is plainly false. So neither \(W1\) nor \(W2\) are injective, according to our definition. Note that the partiality of \(W1\) and \(W2\) is critical for the failure case to occur.

Awkward Case 3: infinite types

Our last tricky case is exemplified by Z here:

\[
\begin{align*}
type family & Z \ a = r | r \to a \\
type instance & Z \ [a] = (a, a) \\
type instance & Z \ (Maybe \ b) = (b, [b])
\end{align*}
\]

Quick: is \(Z\) injective? Are there any types \(s\) and \(t\) for which \(Z \ [t] \sim Z \ (Maybe \ s)\)? Well, by reducing both sides of this equality that would require \((t, t) \sim (s, [s])\). Is that possible? You might think not – after all, the two types do not unify. But consider \(G\), below:

\[
\begin{align*}
type family & G \ a \\
type instance & G \ a = [G \ a]
\end{align*}
\]

(Whether or not \(G\) is injective is irrelevant.) Now choose \(t = s = G \ Int\). We have \(Z \ [G \ Int] \sim (G \ Int, G \ Int) \sim (G \ Int, [G \ Int]) \sim Z \ (Maybe \ (G \ Int))\). Now use Definition 1 on the first and last of this chain of equalities, to deduce \([G \ Int] \sim Maybe \ (G \ Int)\), which is unsound. Indeed \((t, t) \sim (s, [s])\) holds! And so \(Z\) is not injective, according to Definition 1.

One reasonable way to fix this is to guarantee that all type-level functions are terminating, so that we cannot generate an infinite type like \(G \ Int\). However, at the term level Haskell allows potentially non-terminating functions because the burden of proving termination is too heavy. Instead, programmers routinely accept that if they write an infinite loop the program will loop. The type level is similar: the burden of proving termination is heavy, and programmers are willing to accept that the typechecker might loop if they write a type-level loop. Indeed GHC’s language extension UndecidableInstances lifts termination checking for type-class and type family instances for this very reason.

The difference with injectivity is that the type checker might terminate, but generate an unsound program, and that is unacceptable. As long as GHC accepts potentially non-terminating type families, the possibility of such a disaster is real, and we must guard against it.

4.2 The injectivity check

Equipped with these intuitions, we can give the following injectivity-check algorithm:

Definition 2 (Injectivity check). A type family \(F\) is \(n\)-injective iff

1. For every equation \(F \ \tau = \tau\):
   - \(a\) \(\tau\) is not a type family application, and
   - \(b\) \(\tau\) (for some type variable \(a\)), then \(\tau = a\) (that is, the list \(\tau\) consists of just one element, \(a\)).
2. Every pair of equations \(F \ \tau_i = \tau_i\) and \(F \ \tau_j = \tau_j\) (including \(i = j\)) is pairwise-\(n\)-injective.

Clause 2 compares equations pairwise. Here is the intuition, which we will make precise in subsequent sections:

Definition 3 (Intuitive pairwise check). Two equations are pairwise-\(n\)-injective if, when the RHSs are the same, then the \(n\)’th argument on the left hand sides are also the same.

For open type families, we must perform this pairwise-injectivity check to all pairs of type instance declarations in the program, even though they may be scattered over many modules. This is nothing new: the same holds of the check that equations are compatible.

Clause 1 deals with Awkward Case 2, by rejecting any type family with an equation whose RHS is a bare type variable or function call. This restriction is barely noticeable in practice, because any equation rejected by Clause 1 would also be rejected by Clause 2, if there was more than one equation. That leaves only single-equation families, such as

\[
\begin{align*}
type instance & F \ a = G \ a
\end{align*}
\]

which might as well be done with a type synonym. The sole exception are equations of the form \(F \ a = a\), a useful fallback case for a closed type family. We allow this as a special case; hence 1b.

Notice that Condition 1 permits a RHS that is headed by a type variable or function call; e.g. \(F \ (T \ a \ b) = a \ b\) or \(F \ [a] = G \ a \ Int\), where \(G\) has an arity of 1.

4.2.1 Unifying RHSs

In the intuitive injectivity check above, we check if two RHSs are the same. However, type families are, of course, parameterized over variables, so the “sameness” check must really mean unification. For example:
We can always make them disjoint by alpha-conversion. It would be terribly wrong to conclude that \( G_1 \) is injective, just because \( a \) and \( (b, b) \) are not syntactically identical.

Unifying the RHSs will, upon success, yield a substitution. We want to apply that substitution to the LHSs, to see if they become syntactically identical. For example, consider:

\[
\text{type family } G_2 \ a \ b \ = \ r \ | \ r \to a \\
\text{type instance } G_2 \ a \ \text{Bool} \ = \ (a, \text{a}) \\
\text{type instance } G_2 \ b \ \text{Bool} \ = \ (b, \text{Bool})
\]

Unifying the RHSs yields a most-general substitution that sets both \( a \) and \( b \) to \text{Bool}. Under this substitution, the LHSs are the same, and thus \( G_2 \) is injective.

We must be careful about variable names however. Consider \( G_3 \):

\[
\text{type family } G_3 \ a \ b \ = \ r \ | \ r \to b \\
\text{type instance } G_3 \ a \ \text{Int} \ = \ (a, \text{Int}) \\
\text{type instance } G_3 \ b \ \text{Bool} \ = \ (\text{Bool}, \text{b})
\]

This function is not injective; both \( G_3 \ \text{Bool} \ \text{Int} \) and \( G_3 \ \text{Int} \ \text{Bool} \) reduce to \( (\text{Bool}, \text{Int}) \). But the RHSs, as stated, do not unify: the unification algorithm will try to set \( a \) to both \text{Bool} and \text{Int}. The solution is simple: freshen type variables, so that the sets of variables in the equations being compared are disjoint. In this example, if we freshen the \( a \) in the second equation to \( b \), we get a unifying substitution \( [a \mapsto \text{Bool}, b \mapsto \text{Int}] \), and since the LHSs do not coincide under that substitution, we conclude that \( G_3 \) is not injective.

Conveniently, freshening variables and unifying allows us to cover one other corner case, exemplified in \( G_4 \):

\[
\text{type family } G_4 \ a \ b \ = \ r \ | \ r \to a b \\
\text{type instance } G_4 \ a \ [\text{b}] \ = \ [a]
\]

The type family \( G_4 \) is not injective in its second argument, and we can see that by comparing the equation against itself; that is, when we say “every pair of equations” in Definition 3 we include the pair of an equation with itself. When comparing \( G_4 \)’s single equation with itself, variable freshening means that we effectively compare:

\[
\text{type instance } G_4 \ a \ b \ 1 \ = \ [a l b] \\
\text{type instance } G_4 \ a \ b \ 2 \ = \ [a 2]
\]

The unifying substitution can be \( [a l \mapsto a 2] \). Applying this to the LHSs still yields a conflict \( b l \neq b 2 \), and \( G_4 \) is (rightly) discovered to be non-injective. Summing this all together, we can refine our intuitive pairwise check as follows:

**Definition 4** (Unsound pairwise check). Two equations \( F \ \pi_i = \tau_i \) and \( F \ \pi_j = \tau_j \), whose variables are disjoint\(^6\), are pairwise-\(n\)-injective iff either

1. Their RHSs \( \tau_i \) and \( \tau_j \) fail to unify, or
2. Their RHSs \( \tau_i \) and \( \tau_j \) unify with substitution \( \theta \), and \( \theta(\pi_i, a) = \theta(\pi_j, a) \).

Alas, as we saw in Awkward Case 1 (Section 4.1), if the RHS of a type instance can mention a type family, this test is unsound. We explain and tackle that problem next.

### 4.2.2 Type families on the RHS
If the RHS of a type instance can mention a type family, classical unification is not enough. Consider this example:

\[
U(a, \tau) = U(\theta(a), \tau) \quad a \in \text{dom}(\theta) \\
U(a, \tau) = \text{Just } \theta \quad a \in \text{ftv}(\tau) \\
U(a, \tau) = \text{Just } ((a \mapsto \theta(\tau)) \circ \theta) \quad a \notin \text{ftv}(\tau) \\
U(\tau, a) = U(a, \tau) \\
U(\sigma, \tau_1, \tau_2) = U(\sigma_1, \tau_1) \triangleright= U(\sigma_2, \tau_2) \\
U(H, H) = \text{Just } \theta \\
U(F \ \pi_1, F \ \pi_2) = U(\sigma_1, \tau_1) \triangleright= F \text{-injective} \\
\ldots \\
U(\sigma_1, \tau_1) \triangleright= \text{Fj-injective} \\
U(F \ \pi_1, \tau) = \text{Just } \theta \\
U(\tau, F \ \pi_1) = \text{Just } \theta \\
U(\sigma, \tau) = \text{Nothing}
\]

**Just \( \triangleright= k \ = \ k \ \theta \)
Nothing \( \triangleright= k = \text{Nothing} \)

**Figure 2.** Pre-unification algorithm \( U \).

\[
\text{type family } G_5 \ a \ = \ r \ | \ r \to a \\
\text{type instance } G_5 \ a \ = \ [G \ a] \\
\text{type instance } G_5 \ \text{Int} \ = \ [\text{Bool}]
\]

Here, \( G \) is some other type family, known to be injective. When comparing these equations, the RHSs do not unify under the classical definition of unification (i.e. there is no unifying substitution). Therefore, under Definition 4, \( G_5 \) would be accepted as injective. However, this is wrong: we might have \( G \ \text{Int} = \text{Bool} \), in which case \( G_5 \) is plainly not injective.

To fix this problem, we need a variant of the unification algorithm that treats a type family application as potentially unifiable with any other type. Algorithm \( U(\sigma, \tau) \) is defined in Figure 2. It takes types \( \sigma \) and \( \tau \) and a substitution \( \theta \), and returns one of two possible outcomes: \text{Nothing}, or \text{Just } \phi, where \( \phi \) extends \( \theta \). We say that \( \phi \) extends \( \theta \) iff there is a (possibly empty) \( \theta' \) such that \( \phi = \theta' \circ \theta \).

The definition is similar to that of classical unification except:

- Equations (8) and (9) deal with the case of a type-function application; it immediately succeeds without extending the substitution.
- Equation (7) allows \( U \) to recurse into the injective arguments of a type-function application.
- Equation (2) would fail in classical unification (an “occurs check”); \( U \) succeeds immediately, but without extending the substitution. We discuss this case in Section 4.2.3.

We often abbreviate \( U(\sigma, \tau) \perp \) as just \( U(\sigma, \tau) \), where \( \perp \) is the empty substitution.

Algorithm \( U \) has the following two properties:

- If \( U(\sigma, \tau) = \text{Nothing} \), then \( \sigma \) and \( \tau \) are definitely not unifiable, regardless of any type-function reductions\(^7\). For example \( U(\text{Int}, \text{Maybe } a) = \text{Nothing} \), because the rigid structure (here \text{Int}, \text{Maybe}) guarantees that they are distinct types, regardless of any substitution for \( a \).
- If \( U(\sigma, \tau) = \text{Just } \theta \), then it is possible (but not guaranteed) that, some substitution \( \phi \) might make \( \sigma \) and \( \tau \) equal; that is: \( \phi(\sigma) \sim \phi(\tau) \). For example \( U(F \ a, \text{Int}) = \text{Just } \perp \) because perhaps when \( a = \text{Bool} \) we might have a family instance \( F \ \text{Bool} = \text{Int} \).

\(^6\) We can always make them disjoint by alpha-conversion.

\(^7\) Readers may be familiar with apartness from previous work (Eisenberg et al. 2014). To prove the soundness of our injectivity check, we need \( U(\sigma, \tau) = \text{Nothing} \) to imply that \( \sigma \) and \( \tau \) are apart.
However, it is always the case that such a \( \sigma \) extends \( \theta \). Intuitively, \( \theta \) embodies all the information that \( U \) can discover with certainty. We say that \( \theta \) is a pre-unifier of \( \sigma \) and \( \tau \) and we call \( U \) a pre-unification algorithm.

These properties are just what we need to refine previous definition of unsound pairwise check:

**Definition 5** (Pairwise injectivity with pre-unification). A pair of equations \( F \bar{\sigma}_i = \tau_i \) and \( F \bar{\sigma}_j = \tau_j \), whose variables are disjoint, are pairwise-\( n \)-injective iff either

1. \( U(\tau_i, \tau_j) = \text{Nothing} \), or
2. \( U(\tau_i, \tau_j) = \text{Just } \theta \), and \( \theta(\sigma_{i,n}) = \theta(\sigma_{j,n}) \).

As an example, consider \( G5 \) above. Applying the pairwise injectivity with \( U \) test to the two right-hand sides, we find \( U([G \ a], [\text{Bool}]) = \text{Just } \varnothing \), because \( U \) immediately returns when it encounters the call \( G \ a \). That substitution does not make the LHSs identical, so \( G5 \) is rightly rejected as non-injective.

Now consider this definition:

*type family* \( G6 \ a = r \mid r \rightarrow a \)
*type instance* \( G6 \ [a] = [G \ a] \) -- (1)
*type instance* \( G6 \ \text{Bool} = \text{Int} \) -- (2)

Obviously, RHSs of equations (1) and (2) don’t unify. Indeed, calling \( U([G \ a], \text{Int}) \) yields \( \text{Nothing} \) and so the pair (1,2) is pairwise-injective. But the injectivity of \( G6 \) really depends on the injectivity of \( G \): \( G6 \) is injective iff \( G \) is injective. We discover this by performing pairwise test of equation (1) with itself (after freshening). This yields \( U(G \ a, G \ a') \). If \( G \) is injective \( U \) succeeds returning a substitution \( [a \mapsto a'] \) that makes the LHSs identical, so the pair (1,1) is pairwise-injective. If \( G \) is not injective, \( U \) still succeeds, but this time with the empty substitution, so the LHSs do not become identical; so (1,1) would not be pairwise-injective, and \( G6 \) would violate its injectivity condition.

This test is compositional: we can check each definition separately, assuming that the declared injectivity of other definitions holds. In the case of recursive functions, we assume that the declared injectivity holds of calls to the function in its own RHS; and check that, under that assumption, the claimed injectivity holds.

### 4.2.3 Dealing with infinity

Our pre-unification algorithm also deals with Awkward Case 3 in Section 4.1, repeated here:

*type family* \( Z \ a = r \mid r \rightarrow a \)
*type instance* \( Z \ [a] = (a, a) \)
*type instance* \( Z \ (\text{Maybe } b) = (b, [b]) \)

Classical unification would erroneously declare the RHSs as distinct but, as we saw in Section 4.1, there is a substitution which makes them equal. That is the reason for equation (2) in Figure 2: it conservatively refrains from declaring the types definitely-different, and instead succeeds without extending the substitution. Thus \( U((a, a), (b, [b])) \) returns the substitution \( [a \mapsto b] \) but since that doesn’t make the LHSs equal \( Z \) is rejected as non-injective.

### 4.2.4 Closed type families

Consider this example of a closed type family:

*type family* \( G7 \ a = r \mid r \rightarrow a \ where \)
\[
\begin{align*}
G7 \ \text{Int} &= \text{Bool} \\
G7 \ \text{Bool} &= \text{Int} \\
G7 \ a &= a
\end{align*}
\]

The type family \( G7 \) is injective, and we would like to recognize it as such. A straightforward application of the rules we have built up for injectivity will not accept this definition, though. When comparing the first equation against the third, we unify the RHSs, getting the substitution \( [a \mapsto \text{Bool}] \). We apply this to the LHSs and compare \( \text{Int} \) with \( \text{Bool} \); these are not equal, and so the pair of equations appears to be a counter-example to injectivity. Yet, something is amiss: the third equation cannot reduce with \( [a \mapsto \text{Bool}] \), since the third equation is shadowed by the second one.

This condition is easy to check for. When checking LHSs with a substitution derived from unifying RHSs, we just make sure that if LHSs are different then at least one of the two equations cannot fire after applying the substitution:

**Definition 6** (Pairwise injectivity). A pair of equations \( F \bar{\sigma}_i = \tau_i \) and \( F \bar{\sigma}_j = \tau_j \), whose variables are disjoint, are pairwise-\( n \)-injective iff either

1. \( U(\tau_i, \tau_j) = \text{Nothing} \), or
2. \( U(\tau_i, \tau_j) = \text{Just } \theta \), and \( \theta(\sigma_{i,n}) = \theta(\sigma_{j,n}) \).

Note that in an open type family, applying a substitution to an equation’s LHS will always yield a form reducible by that equation, so the last two clauses are always false. As a result, Definition 6 works for both open and closed type families.

#### 4.3 Soundness

We have just developed a subtle algorithm for checking injectivity annotations. But is the algorithm sound?

**Property 7** (Soundness). If the injectivity check concludes that \( F \) is \( n \)-injective, then \( F \) is \( n \)-injective, in the sense of Definition 1.

In Appendix C, we prove a slightly weaker variant of the property above, and we conjecture the full property. The change we found necessary was to omit equation (7) from the statement of the pre-unification algorithm \( U \); this equation allows algorithm \( U \) to look under injective type families on the RHS. Without that line, a use of an injective type family in an RHS is treated as if any other type family. Such a modified pre-unification algorithm labels fewer functions as injective. For example, it would reject

*type family* \( F \ a = r \mid r \rightarrow a \)
*type instance* \( F \ a = \text{Maybe } (G \ a) \)

even if \( G \) were known to be injective.

The full check is quite hard to characterize: what property precisely, holds of a substitution produced by \( U(\tau, \sigma) \)? We have said that this substitution is a pre-unifier of \( \tau \) and \( \sigma \), but that fact alone is not enough to prove soundness. We leave a full proof as future work.

#### 4.4 Completeness

The injectivity check described here is easily seen to be incomplete. For example, consider the following collection of definitions:

*type family* \( F \ a = r \mid r \rightarrow a \)
*type instance* \( F \ (\text{Maybe } a) = G \ a \)
*type instance* \( F \ [a] = H \ a \)
*type family* \( G \ a = r \mid r \rightarrow a \)
*type instance* \( G \ a = \text{Maybe } a \)
*type family* \( H \ a = r \mid r \rightarrow a \)
*type instance* \( H \ a = [a] \)

The type function \( F \) is a glorified identity function, defined only over lists and \( \text{Maybe} \). It is injective. Yet, our check will reject it, because it does not reason about the fact that the ranges of \( G \) and \( H \) are disjoint. Indeed, as argued at the beginning of Section 4, any compositional algorithm will suffer from this problem.
Yet, we would like some completeness property. We settle for this one:

**Property 8 (Completeness).** Suppose a type family $F$ has equations such that for all right-hand sides $\tau$:

- $\tau$ is type-family-free,
- $\tau$ has no repeated use of a variable, and
- $\tau$ is not a bare variable.

If $F$ is $n$-injective, then the injectivity check will conclude that $F$ is $n$-injective.

Under these conditions, the pairwise injectivity check becomes the much simpler unifying pairwise check of Definition 4, which is enough to guarantee completeness. Note that the conditions mean that Algorithm $U$ operates as a classical unification algorithm (effectively eliminating equations (2), (7), (8), and (9) from the definition of $U$) and that we no longer have to worry about the single-equation checks motivated by Awkward Case 2 (clause 1 of Definition 2). The proof appears in Appendix D.

### 5. Exploiting injectivity

It is all very well knowing that a function is injective, but how is this fact useful? There are two separate ways in which injectivity can be exploited:

**Improvement** guides the type inference engine, by helping it to fix the values of as-yet-unknown unification variables. Improvement comes in two parts: improvement between “wanted” constraints (Section 5.1) and improvement between wanted constraints and top-level type-family equations (Section 5.2). These improvement rules correspond directly to similar rules for functional dependencies, as we discuss in Section 7.

**Decomposition** of “given” constraints enriches the set of available proofs, and hence makes more programs typeable (Section 5.3). Unlike improvement, which affects only inference, decomposition requires a small change to GHC’s explicitly typed intermediate language, System FC.

#### 5.1 Improvement of wanted constraints

Suppose we are given these two definitions:

\[
\begin{align*}
  f &:: F \ a \rightarrow Int \\
  g &:: Int \rightarrow F \ b
\end{align*}
\]

Is the call $f\ (g\ 3)$ well typed? Obviously yes, but it is hard for a type inference engine to determine that this is so without knowing about the injectivity of $F$. Suppose we instantiate the call to $f$ with a unification variable $\alpha$, and the call to $g$ with $\beta$. Then we have to prove that $F \alpha \sim F \beta$; we use the term “wanted constraint” for constraints that the inference engine must solve to ensure type safety.

We can certainly solve this constraint if we clairvoyantly unify $\alpha ::= \beta$. But the inference engine only performs unifications that it knows must hold; we say that it performs only guess-free unification (Vytiniotis et al. 2011, Section 3.6). Why? Suppose that (in a larger example) we had this group of three wanted constraints:

\[
F \alpha \sim F \beta \quad \alpha \sim Int \quad \beta \sim Bool
\]

Then the right thing to do would be unify $\alpha ::= Int$ and $\beta ::= Bool$, and hope that $F\ Int$ and $F\ Bool$ reduce to the same thing. Instead unifying $\alpha ::= \beta$ would wrongly lead to failure.

So, faced with the constraint $F \alpha \sim F \beta$, the inference engine does not in general unify $\alpha ::= \beta$; so the constraint $F \alpha \sim F \beta$ is not solved, and hence $f\ (g\ 3)$ will be rejected. But if we knew that $F$ was injective, we can unify $\alpha ::= \beta$ without guessing.

**Improvement** (a term due to Mark Jones (Jones 1995, 2000)) is a process that adds extra “derived” equality constraints that may make some extra unifications apparent, thus allowing inference to proceed further without having to make guesses. In the case of an injective $F$, improvement adds $\alpha \sim \beta$, which the constraint solver can solve by unification. In general, improvement of wanted constraint is extremely simple:

**Definition 9 (Wanted improvement).** Given the wanted constraint $F \sigma \sim F \tau$, add the derived wanted constraint $\sigma_n \sim \tau_n$ for each $n$-injective argument of $F$.

Why is this OK? Because if it is possible to prove the original constraint $F \sigma \sim F \tau$, then (by Definition 1) we will also have a proof of $\sigma_n \sim \tau_n$. So adding $\sigma_n \sim \tau_n$ as a new wanted constraint does not constrain the solution space. Why is it beneficial? Because, as we have seen, it may expose additional guess-free unification opportunities that that solver can exploit.

#### 5.2 Improvement via type family equations

Suppose we have the top-level equation

\[
\text{type instance } F\ [a] = a \rightarrow a
\]

and we are trying to solve a wanted constraint $F\ \alpha \sim (Int \rightarrow Int)$, where $\alpha$ is a unification variable. The top-level equation is shorthand for a family of equalities, namely its instances under substitutions for $a$, including $F\ [Int] \sim (Int \rightarrow Int)$. Now we can use the same approach as in the previous section to add a derived equality $\alpha \sim [Int]$. That in turn will let the constraint solver unify $\alpha ::= [Int]$, and thence solve the wanted constraint. So the idea is to match the RHS of the equation against the constraint and, if the match succeeds add a derived equality for each injective argument.

Matters are more interesting when there is a function call on the RHS of the top-level equation. For example, consider $G6$ from Section 4.2.2, when $G$ is injective:

- **type family** $G6\ a = r \mid r \rightarrow a$
- **type instance** $G6\ [a] = [G\ a]$
- **type instance** $G6\ Bool = Int$

Suppose we have a wanted constraint $G6\ \alpha \sim [Int]$. Does the RHS of the equation, $[G\ a]$, match the RHS of the constraint $[Int]$? Apparently not; but this is certainly the only equation for $G6$ that can apply (because of injectivity). So the argument $\alpha$ must be a list, even if we don’t know what its element type is. So we can produce a new derived constraint $\alpha \sim [\beta]$, where $\beta$ is a fresh unification variable. This expresses some information about $\alpha$ (namely that it must be a list type), but not all (the fresh $\beta$ leaves open what the list element type might be). We might call this partial improvement.

Partial improvement is very useful indeed! We can now unify $\alpha ::= [\beta]$, so the wanted constraint becomes $G6\ [\beta] \sim [Int]$. Now $G6$ can take a step, yielding $[G\ \beta] \sim [Int]$, and decompose to get $G\ \beta \sim Int$. Now the process may repeat, with $G$ instead of $G6$. The crucial points are that (a) the matching step, like the pre-unification algorithm $U$, behaves specially for type-family calls; and (b) we instantiate any unmatched variables with fresh unification variables. More formally:

**Definition 10 (Top-level improvement).** Given:

- an equation $i$ of type family $F$, $F\ \overline{\sigma_i} = \tau_i$, and
- a wanted constraint $F\ \overline{\sigma} \sim \tau_0$

such that

- $M(\tau_i, \tau_0) = \text{Just } \theta$, and
- $F\ \theta(\overline{\sigma})$ can reduce via equation $i$
then define \( \theta' \) by extending \( \theta \) with \( a \mapsto \alpha \), for every \( a \in \Sigma \), that is not in \( \text{dom}(\theta) \), where \( \alpha \) is a fresh unification variable; and add a derived constraint \( \theta'(\sigma_i n) \sim \sigma_{0n} \), for every \( n \)-injective argument of \( F \).

Here \( M \) is defined just like \( U \) in Figure 2, except lacking equations (4) and (9). That is, \( M \) does one-way matching rather than two-way unification. (We assume that the variables of the two arguments to \( M \) do not overlap.)

5.3 Decomposing given equalities

Consider the following function, where \( F \) is an injective type family:

\[
\begin{align*}
\text{fid} &:: (F \ a \sim F \ b) \implies a \sim b \\
\text{fid} &x = x
\end{align*}
\]

Should that type-check? Absolutely. We assume that \( F \ a \sim F \ b \), and by injectivity (Definition 1), we know that \( a \sim b \). But, arranging for GHC to compile this requires a change to System FC. Here in FC, all type abstractions, applications, and casts are explicit. FC code uses a proof term, or coercion, that witnesses the truth of each equality constraint. In FC, \( \text{fid} \) takes an argument coercion \( e :: F \ a \sim F \ b \), but needs a coercion of type \( a \sim b \) to cast \( x :: a \) to the desired result type \( b \). The FC code for \( \text{fid} \) looks like this:

\[
\begin{align*}
\text{fid} &:: \forall \ a \ b. \ (F \ a \sim F \ b) \implies a \sim b \\
\text{fid} &= \Lambda a \ b \to \lambda (c :: F \ a \sim F \ b) \ (x :: a) \to x \triangleright (\text{nth}^0 \ c)
\end{align*}
\]

The coercion \( \text{nth}^0 \ c \) is a proof term witnessing \( a \sim b \); using \( \text{nth} \) to decompose a type family application is the extension required to FC, as we discuss next.

5.3.1 Adding type family injectivity to FC

To a first approximation, System FC is Girard’s System F, enhanced with equality coercions. That is, there is a form of expression \( e \triangleright \gamma \) that casts \( e \) to have a new type, as shown by the following typing rule:

\[
\frac{
\Gamma \vdash e : \tau_1 \\
\Gamma \vdash \gamma : \tau_1 \sim \tau_2
}{
\Gamma \vdash e \triangleright \gamma : \tau_2} \quad \text{TM\_CAST}
\]

The unusual typing judgement \( \Gamma \vdash \gamma : \tau_1 \sim \tau_2 \) says that \( \gamma \) is a proof, or witness, that type \( \tau_1 \) equals type \( \tau_2 \).

Coercions \( \gamma \) have a variety of forms, witnessing the properties of equality required from System FC. For example, there are forms witnessing reflexivity, symmetry, and transitivity, as well as congruence of application; the latter allows us to prove that types \( \tau_1 \sim \tau_2 \) and \( \sigma_1 \sim \sigma_2 \) are equal from proofs that \( \tau_1 \sim \sigma_1 \) and \( \tau_2 \sim \sigma_2 \).

The coercion form that concerns us here is the one that witnesses injectivity. In previous versions of FC, the rule looked thus:

\[
\frac{
\Gamma \vdash \gamma : H \sigma \sim H \sigma' \\
\Gamma \vdash \text{nth}^1 \gamma : \tau_1 \sim \sigma_i
}{
\Gamma \vdash \text{CO\_NTH}
}
\]

In this rule, \( H \) is a type constant (such as \( \text{Maybe} \) or \( \text{Either} \)), all of which are considered to be injective in Haskell. The coercion \( \text{nth}^1 \gamma \) witnesses this injectivity by proving equality among arguments from the equality of the applied datatype constructor.

To witness injective type families, we must add a new rule as follows:

\[
\frac{
\Gamma \vdash F \sim F' \\
\text{F is } i\text{-injective}
}{
\Gamma \vdash \text{CO\_NTH\_TyFAM}
\]

In this rule, \( F \) is a type family. We can now extract an equality among arguments from the equality proof of the type family applications.

\[
\begin{align*}
U(a :: \kappa_1, \tau :: \kappa_2) \theta &= U(a :: \kappa_1, \kappa_2) \theta \\
(a :: \kappa_1, \tau :: \kappa_2)\theta &= U(a :: \kappa_1, \kappa_2) [(a \mapsto \theta(\kappa)] \circ \theta) \\
\end{align*}
\]

Figure 3. Modified equations (2) and (3) from Figure 2 that make the pre-unification algorithm \( U \) kind-aware.

5.3.2 Soundness of type family injectivity

Having changed GHC’s core language, we now have the burden of proving our change to be type safe. The key lemma we must consider is the consistency lemma. Briefly, the consistency lemma states that, in a context with no equality assumptions, it is impossible to prove propositions like \( \text{Int} \sim \text{Bool} \), or \( (a \rightarrow b) \sim \text{IO} \). With the consistency lemma in hand, the rest of the proof of type safety would proceed as it has in previous publications, for example Breitner et al. (2014a).

Even stating the key lemmas formally would require diving deeper into System FC than is necessary here; the lemmas and their proofs appear in Appendix C.

5.4 Partial type functions

Both open and closed type families may be partial; that is, defined on only part of their domain. For example, consider this definition for an injective function \( F \):

\[
\begin{align*}
\text{type family} \quad F \ a &= r \mid r \rightarrow a \\
\text{type instance} \quad F \ 	ext{Int} &= \text{Bool} \\
\text{type instance} \quad F \ [a] &= a \rightarrow a
\end{align*}
\]

The type \( F \ \text{Char} \) is equal to \( \text{Char} \rightarrow \text{Char} \), by the second instance above; but \( F \ 	ext{Bool} \) is equal only to itself since it matches no equation. Nevertheless, \( F \) passes our injectivity test (Section 4). You might worry that partiality complicates our story for injectivity. If we had a wanted constraint \( F \ 	ext{Bool} \sim F \ 	ext{Char} \), our improvement rules would add the derived equality \( \text{Bool} \sim \text{Char} \), which is manifestly insoluble. But nothing has gone wrong: the original wanted constraint was also insoluble (that is, we could not cough up a coercion that witnesses it), so all the derived constraint has done is to make that insolubility more stark.

In short, the fact that type functions can be partial does not gum up the works for type inference.

6. Injectivity in the presence of kind polymorphism

Within GHC, kind variables are treated like type variables: type family arguments can include both kinds and types. Thus type families can be injective not only in type arguments but also in kind arguments. To achieve this we allow kind variables to be mentioned in the injectivity condition, just like type variables. Moreover, if a user lists a type variable \( b \) as injective, then all kind variables mentioned in \( b \)'s kind are also marked as injective. For example:

\[
\begin{align*}
\text{type family} \quad G \ (a :: \kappa 1) (b :: \kappa 2) (c :: \kappa 1) &= \ (r :: \kappa 3) \mid r \rightarrow b \ k1 \\
\text{type instance} \quad G \ 	ext{Maybe} \ 	ext{Int} \ (\text{Either} \ 	ext{Bool}) &= \ 	ext{Char} \\
\text{type instance} \quad G \ 	ext{IO} \ 	ext{Int} \ [\ ] &= \ 	ext{Char} \\
\text{type instance} \quad G \ 	ext{Either} \ 	ext{Bool} \ (\rightarrow) &= \ 	ext{Maybe}
\end{align*}
\]

The injectivity annotation on \( G \) states that it is injective in \( b \) — and therefore also in \( b \)'s kind \( k2 \) — as well as kind \( k1 \), which is the kind of both \( a \) and \( c \). We could even declare \( k3 \) as injective — the return kind is also an input argument to a type family.

To support injectivity in kinds our pre-unification algorithm \( U \) needs a small adjustment to make it kind-aware — see modified...
equations (2) and (3) in Figure 3. Other definitions described in Sections 4 and 5 remain unchanged.

In Haskell source, in contrast to within GHC, kind arguments are treated quite separately from type arguments. Types are always explicit, while kinds are always implicit. This can lead to some surprising behaviour:

\[
\text{type family } P \ (a :: k0) = (r :: k1) \mid r \to a \\
\text{type instance } P \ [\ ] = \cdot
\]

At first glance, \(P\) might look injective, yet it is not. Injectivity in \(a\) means injectivity also in \(k0\). But the argument \(a\) and result \(r\) can have different kinds and so \(k0\) is not determined by \(r\). This becomes obvious if we write kind arguments explicitly using a hypothetical syntax, where the kind arguments are written in braces:

\[
\text{type instance } P \ \{k0\} \ \{k1\} \ \{[\ ]\} = \{[\ ]\} \ {k1}\)
\]

The syntax \([\ ]\) \(\{k\}\) indicates an empty type-level list, holding elements of kind \(k\). It is now clear that \(k0\) is not mentioned anywhere in the RHS, and thus we cannot accept it as injective.

### 7. Functional dependencies

Injective type families are very closely related to type classes with functional dependencies (Jones 2000), which have been part of GHC for many years. Like injectivity, functional dependencies appear quite simple, but are remarkably subtle in practice (Sulzmann et al. 2007).

Functional dependencies express a type level function as a relation. For example, here is a type-level function \(F\) expressed using functional dependencies (on the left) and type families (on the right):

\[
\begin{align*}
\text{class } & F \ a \ r \mid a \to r \\
\text{instance } & F \ [a] \ (Maybe\ a) \text{ type family } F \ a = r \\
& \text{instance } F \ Int\Bool \quad \text{type instance } F \ [Int] = Bool
\end{align*}
\]

\[
\begin{align*}
& f :: F\ a\ r \Rightarrow a \to r \\
& f :: a \to F\ a
\end{align*}
\]

To express that \(F\) is injective using functional dependencies, one adds an additional dependency:

\[
\begin{align*}
\text{class } & F \ a \ r \mid a \to r, r \to a
\end{align*}
\]

This syntax motivates our choice of syntax for injectivity annotations (Section 3.2).

Our injectivity check mirrors precisely the consistency checks necessary for functional dependencies. In Section 4.2.2 we discussed the issues that arise when a type family call appears in the RHS of a type instance, such as:

\[
\begin{align*}
\text{type instance } & F \ [a] = \{G\ a\}
\end{align*}
\]

Precisely the same set of issues arises with functional dependencies, where the instance declaration would look like:

\[
\begin{align*}
\text{instance } & G \ a\ rg \Rightarrow F\ [a]\ [rg]
\end{align*}
\]

This instance declaration would fail the coverage condition of (Jones 2000); in effect, Jones does not allow function calls on the RHS. This restriction was lifted by Sulzmann et al., via the liberal coverage condition (Sulzmann et al., 2007), in essentially the same way that we do.

Using “improvement” to guide type inference (Section 5), including the partial improvement of Section 5.2, was suggested by Mark Jones for his system of qualified types (Jones 1995), and was absolutely essential for effective type inference with functional dependencies (Jones 2000). Indeed, the improvement rules of Section 5 correspond precisely to the improvement rules for functional dependencies (Sulzmann et al. 2007).

#### 7.1 Advantages of type families

A superficial but important advantage of type families is simply that they use functional, rather than relational, notation, thus allowing programmers to use same programming style at the type level that they use at the term level. Recognizing this, Jones also proposes some syntactic sugar to make the surface syntax of functional dependencies more function-like (Jones 2008). Syntactic sugar always carries a price, of course: since the actual types will have quantified constraints that are not visible to the programmer, the compiler has to work hard to express error messages, inferred types, and so on, in the form that the programmer expects.

A more substantial difference is that type families are fully integrated into System FC, GHC’s typed intermediate language. Consider, for example:

\[
\begin{align*}
\text{type instance } & F \ Int = Bool \\
& \text{data } T \ a \ where \ {\{\text{MkT} :: F\ a \Rightarrow T\ a\}}
\end{align*}
\]

\[
\begin{align*}
& f :: T\ Int \Rightarrow Bool \\
& f\ (\text{MkT}\ x) = \text{not}\ x
\end{align*}
\]

This typechecks fine. But with functional dependencies we would write:

\[
\begin{align*}
& \text{class } F \ a \ r \mid a \to r \\
& \text{instance } F \ Int\ Bool \\
& \text{data } T \ a \ where \ {\{\text{MkT} :: F\ a \Rightarrow r \Rightarrow T\ a\}}
\end{align*}
\]

and now the definition of \(f\) would be rejected because \(r\) is an existentially captured type variable of \(MkT\). One could speculate on a variant of System FC that accommodated functional dependencies, but no such calculus currently exists.

#### 7.2 Advantages of functional dependencies

Functional dependencies make it easy to specify more complex dependencies than mere injectivity. For example\(^8\):

\[
\begin{align*}
& \text{data } Nat = \text{Zero} \mid \text{Succ}\ a \\
& \text{class } Add\ a\ b\ r \mid a\ b \Rightarrow r \Rightarrow a \to b \\
& \text{instance } \text{Add}\ Zero\ b\ a \\
& \text{instance } \text{Add}\ (\text{Succ}\ a)\ b\ (\text{Succ}\ r)
\end{align*}
\]

Note the dependency “\(r\ a \Rightarrow b\)”, which says that the result and first argument (but not the result alone) are enough to fix the second argument. This dependency leads to an improvement rule: from the wanted constraint \((\text{Add}\ s\ t1) \sim (\text{Add}\ s\ t2)\), add the derived equality \(t1 \sim t2\).

Our design can similarly be extended, by writing:

\[
\begin{align*}
& \text{type family } \text{AddTF}\ a\ b\ r \mid r \Rightarrow a \to b \text{ where} \\
& \text{AddTF}\ Zero\ b\ a \\
& \text{AddTF}\ (\text{Succ}\ a)\ b\ (\text{Succ}\ a\ b)
\end{align*}
\]

The check that the injectivity annotation is sound is a straightforward extension of Definitions 2 and 6, and the improvement rule is just as for functional dependencies. (In the extended rule, Clause 1 of Definition 2 holds if any of the types on the left of the dependency arrow are not a bare variable or function application.) However, this remains as future work: we have not yet extended the metatheory or implementation to accommodate it.

---

\(^8\) Here \(Nat\) is being used as a kind, using the \(\text{DataKinds}\) extension (Yorgey et al. 2012).
7.3 Summary

So, are type families merely functional dependencies in a different guise? At a fundamental level, yes: they both address a similar question in a similar way. But it is always illuminating to revisit an old landscape from a new direction, and we believe that is very much the case here, especially since the landscape of functional dependencies is itself very subtle (Sulzmann et al. 2007). Understanding the connection better is our main item of further work. For example:

- Adding richer functional dependencies to type families (Section 7.2) is an early priority.
- Could we take advantage of the metatheory of functional dependencies to illuminate that of type families; or vice versa?
- What would a version of System FC that truly accommodated functional dependencies look like?
- Could closed type families move beyond injectivity and functional dependencies by applying closed-world reasoning that derives solutions of arbitrary equalities, provided a unique solution exists? Consider this example:

  \[
  \text{type family } J \ a \ where \\
  J \ \text{Int} = \text{Char} \\
  J \ \text{Bool} = \text{Char} \\
  J \ \text{Double} = \text{Float}
  \]

  One might reasonably expect that if we wish to prove \((J \ a \sim \text{Float})\), we will simplify to \((a \sim \text{Double})\). Yet GHC does not do this as neither injectivity nor functional dependencies can discover this solution.

8. Other related work

8.1 Injectivity for the Utrecht Haskell Compiler

Implementing injective type families for the Utrecht Haskell Compiler was proposed by Serrano Mena (2014). In private correspondence Serrano Mena informed us that these ideas were not developed further or implemented. Thus, to our best knowledge, our work is the first theoretical and practical treatment of injectivity for Haskell.

8.2 Injectivity in other languages

The Agda (Norell 2007) compiler is able to infer head injectivity\(^{10}\), a notion weaker than the injectivity presented in this paper. For a function \(f\), if the right-hand sides of all clauses of \(f\) immediately disunify, then \(f\) is called head-injective or constructor-headed. "Immediately disunify" means that the outer-most constructors in the RHSs are distinct. Knowledge that a function is head-injective can then be used to generate improvements in the same way it is used in our solution. Our solution is more powerful: it recurs over identical constructors, allows type families in RHSs, and permits declaring injectivity only in some arguments.

Other dependently-typed languages like Coq (The Coq development team 2014) or Idris (Brady 2013) do not provide any special way of declaring that a function is injective. In these languages the user can prove injectivity of a function using mechanisms provided by the language (e.g. tactics) and appeal to injectivity explicitly whenever this property is required to make progress during type checking. We believe that these languages could benefit from approach developed here – our results should carry over to these other languages nicely.

8.3 Injectivity of term-rewriting systems

Haskell type families share much with traditional term-rewriting systems (TRSs). (For some general background on TRSs, see Baader and Nipkow (1998).) In particular, Haskell type family reduction forms a deterministic constructor term-rewriting system. There has been some work done on checking TRSs for injectivity, for example that of Nishida and Sakai (2010). Their work appears to be the state-of-the-art in the term-rewriting community. Although a close technical comparison of our work to theirs is beyond the scope of this paper, Nishida and Sakai restrict their injectivity analysis to total, terminating systems. Our work also considers partial and non-terminating functions.

9. Conclusion

With this work, we give users a new tool for more expressive type-level programming, and one that solves practical problems arising in the wild (Section 2). It fills out a missing corner of GHC’s support for type-level programming, and gives an interesting new perspective on functional dependencies (Section 7.3).

Our compositional approach for determining injectivity of functions defined by pattern matching may be of more general utility.

Acknowledgements We gratefully acknowledge helpful feedback from Iavor Diatchki, Martin Sulzmann, Dimitrios Vytinioti, and Stephanie Weirich.

References

N. Nishida and M. Sakai. Proving injectivity of functions via program inversion in term rewriting. In M. Blume, N. Kobayashi, and G. Vidal,
A. Popularity of selected language extensions for type-level programming

In Section 1 we made a claim that type families are the most popular language extension for type-level programming in Haskell. That claim is based on analysis of Haskell, Haskell’s community package database. We were interested in usage of five language extensions that in our opinion add the most powerful features to type-level language: TypeFamilies, GADTs, FunctionalDependencies, DataKinds and PolyKinds. To measure their popularity we downloaded all packages on Hackage (per list available at https://hackage.haskell.org/packages/). Then we used the grep program to search each package directory for appearances of strings naming the given language extensions. This located language extensions enabled both in .cabal files and with LANGUAGE pragmas. The exact obtained numbers are reported in Table 1.

<table>
<thead>
<tr>
<th>Language extension</th>
<th>no. of using packages</th>
</tr>
</thead>
<tbody>
<tr>
<td>TypeFamilies</td>
<td>1092</td>
</tr>
<tr>
<td>GADTs</td>
<td>612</td>
</tr>
<tr>
<td>FunctionalDependencies</td>
<td>563</td>
</tr>
<tr>
<td>DataKinds</td>
<td>247</td>
</tr>
<tr>
<td>PolyKinds</td>
<td>109</td>
</tr>
</tbody>
</table>

Table 1. Popularity of selected type-level programming language extensions.

Downside of this approach is that it can give false positives by finding strings without considering their context inside the source code. A good example of when this happens is haskell-src-exts package that does not use any of the above extensions but mentions them in the parser source code.

All measurements were conducted on a copy of Hackage obtained on 19th February 2015.

B. An excerpt of System FC

We present an excerpt of System FC here. For the full details, please see previous work, such as Breitner et al. (2014b).

B.1 Grammar

Metavariables:

- \( a, b \): type variables
- \( c \): coercion variables
- \( T \): algebraic datatypes
- \( F \): type functions
- \( C \): type family axioms

Nonterminals:

- \( H \) :: \( (\rightarrow) \) | \( (\Rightarrow) \) | \( (~) \): type constants
- \( \kappa \) :: \( \ast \) | \( \kappa_1 \rightarrow \kappa_2 \): kinds
- \( \tau, \sigma \) :: \( a \) | \( \tau_1 \tau_2 \) | \( H \) | \( \forall \alpha, \kappa . \tau \) | \( F \tau \): types
- \( \psi \) :: \( H \) | \( \forall \alpha, \kappa . \tau \) | \( \psi \tau \): value types
- \( \gamma \) :: \( \langle \tau \rangle | \text{sym} \gamma \mid \gamma_1 \vdash \gamma_2 \mid F(\tau) \) | \( \gamma_1 \gamma_2 \mid \forall \alpha, \kappa, \gamma \mid c \mid C(\tau) \) | \( \text{nth}\gamma \mid \text{left} \gamma \mid \text{right} \gamma \mid \gamma \circ \tau \): coeencions
- \( \theta \) :: \( \emptyset \mid \theta, [a \mapsto \tau] \): substitutions
- \( \Gamma \) :: \( \emptyset \mid \theta \mid \alpha, \kappa \mid \ldots \): typing contexts

B.2 The coercion formation rules

\[
\Gamma \vdash \gamma : \phi \\
\Gamma \vdash \tau : \kappa \\
\text{CO_REFL} \\
\Gamma \vdash \langle \tau \rangle : \tau \sim \tau \\
\Gamma \vdash \gamma : \kappa \\
\text{CO_SYM} \\
\Gamma \vdash \text{sym} \gamma : \tau \sim \tau \\
\Gamma \vdash \gamma_1 \sim \gamma_2 \\
\Gamma \vdash \gamma_2 \sim \gamma_1 \\
\text{CO_TRANS} \\
\Gamma \vdash \gamma_1 \vdash \gamma_2 \sim \gamma_1 \\
\Gamma \vdash \text{nth} \gamma : \tau_1 \sim \tau_1 \\
\Gamma \vdash F \tau : \kappa \sim \sigma \\
\Gamma \vdash F(\langle \tau \rangle) : F \tau \sim F \sigma \\
\text{CO_TYFAM} \\
\Gamma \vdash \gamma_1 : \tau_1 \sim \sigma_1 \\
\Gamma \vdash \gamma_2 : \tau_2 \sim \sigma_2 \\
\Gamma \vdash \gamma_1 \gamma_2 : \kappa \\
\Gamma \vdash \gamma_1 \gamma_2 : \tau_1 \gamma_2 \sim \sigma_1 \sigma_2 \\
\Gamma, \alpha, \kappa : \gamma : \tau \sim \sigma \\
\Gamma \vdash \forall \alpha, \kappa, \gamma : \forall \alpha, \kappa \sim \forall \alpha, \kappa, \sigma \\
\text{CO_FORALL} \\
\Gamma \vdash \forall \alpha, \kappa, \gamma : \forall \alpha, \kappa \sim \forall \alpha, \kappa, \sigma \\
\Gamma \vdash c : \tau \sim \sigma \in \mathbb{E} \\
\Gamma \vdash c : \tau \sim \sigma \\
\text{CO_VAR} \\
\Gamma \vdash C(\tau), \sigma_1 \sim \sigma_2 \\
\Gamma \vdash \gamma : \kappa \\
\Gamma \vdash H \tau \sim H \sigma \\
\text{CO_NTH} \\
\Gamma \vdash \gamma : F \tau \sim F \sigma \\
F \text{ is i-injective} \\
\Gamma \vdash \text{nth} \gamma : \tau_1 \sim \sigma_1 \\
\text{CO_NTH_TYFAM} \\
\Gamma \vdash \gamma : \kappa \\
\text{C} : [\pi, \kappa], \sigma_1 \sim \sigma_2 \\
\Gamma \vdash \tau : \kappa \\
\text{CO_AXIOM} \\
\Gamma \vdash C(\tau), \sigma_1 \sim \sigma_2 \\
\Gamma \vdash \gamma : \kappa \\
\Gamma \vdash H \tau \sim H \sigma \\
\text{CO_NTH} \\
\Gamma \vdash \gamma : F \tau \sim F \sigma \\
F \text{ is i-injective} \\
\Gamma \vdash \text{nth} \gamma : \tau_1 \sim \sigma_1 \\
\text{CO_NTH_TYFAM}
\]
C. Proof of type family injectivity soundness

We follow along the structure of the type soundness proof of System FC as written in Breitner et al. (2014b); some passages in this appendix are taken verbatim from that work. (Some passages are also taken verbatim from Eisenberg et al. (2014).) Breitner et al. (2014b) is concerned quite intricately with roles. Fortunately, roles do not interact with type family injectivity, because type families always only operate at nominal roles. We thus omit roles and the distinction between datatypes and newtypes throughout.

**Definition 11** (Type consistency). Two types $\tau_1$ and $\tau_2$ are consistent if, whenever they are both value types:

1. If $\Gamma \vdash \gamma : \tau_1 \sim \tau_2$, then $\Gamma \vdash \gamma : \tau_1 \sim \tau_2$.
2. If $\Gamma \vdash \gamma : \tau_1 \sim \tau_2$, then $\Gamma \vdash \gamma : \tau_1 \sim \tau_2$.

Note that if either $\tau_1$ or $\tau_2$ is not a value type, then they are vacuously consistent.

**Definition 12** (Context consistency). The global context (containing datatype and type family definitions) is consistent if, whenever $\not\vdash \gamma : \tau_1 \sim \tau_2$, $\tau_1$, and $\tau_2$ are consistent.

If the global context is consistent, that means that no coercion exists (in an empty environment) that proves, say, $\text{Int} \sim \text{Bool}$. In order to prove context consistency, we define a type reduction relation $\tau \rightarrow \sigma$, show that the relation preserves value type heads, and then show that any well-typed coercion corresponds to a path in the rewrite relation.

Here is the type rewrite relation:

- $\tau \rightarrow \sigma$ Type reduction
- $\tau \rightarrow \tau$ RED_REFL
- $\tau_1 \sim \tau \rightarrow \tau_1 \sim \tau$ RED_APP
- $\forall \alpha. \tau \sim \sigma \rightarrow \forall \alpha. \tau \sim \sigma$ RED_FORALL
- $F \not\sim \sigma$ RED_TYFAM
- $C : \exists \alpha. \tau \sim \tau_1 \rightarrow \tau_2$ RED_AXIOM

**Definition 13** (Joinability). We say that two types $\tau_1$ and $\tau_2$ are joinable, written $\tau_1 \leftarrow \tau_2$, if there exists $\sigma$ such that $\tau_1 \sim^* \sigma$ and $\tau_2 \sim^* \sigma$.

**Lemma 14** (Rewrite substitution). Let $\pi$ be the free variables in a type $\sigma$. If $\pi \sim \pi'$, then $\sigma[\pi/\pi'] \sim \sigma[\pi'/\pi']$.

**Proof.** Proved as Lemma 26 in Breitner et al. (2014b).

**Lemma 15** (Confluence). If $\tau \sim^* \sigma_1$ and $\tau \sim^* \sigma_2$, then $\sigma_1 \leftarrow \sigma_2$.

**Proof.** Proved as Lemma 32 in Breitner et al. (2014b).

**Lemma 16** (Non-linear patterns). Let $\pi$ be the free variables in a type $\tau$. We require that no type families appear in $\pi$. If, for some $\sigma_1, \tau[\sigma_1/\pi] \sim \tau'$, then there exist $\sigma'$ such that $\tau' \sim^* \tau[\sigma'/\pi]$ and $\sigma \sim^* \sigma'$.

This lemma is very similar to the Pattern lemma (Lemma 29) of Breitner et al. (2014b). It differs in three ways:

1. This lemma does not require each of the $\pi$ to appear only once in $\tau$.
2. It allows the $\sigma$ to multistep to reach the $\sigma'$.
3. And it concludes that $\tau'$ multisteps to $\tau[\sigma'/\pi]$, but these might not equal.

This lemma cannot replace the original Pattern lemma, however, because it is critical in the proof of confluence that $\sigma$ can reach $\sigma'$ in one step. The proof of this lemma, in turn, uses confluence.

**Proof.** We proceed by induction on the structure of $\tau$.

Case $\tau = \alpha$: There is just one free variable ($\alpha$), and thus just one type $\tau$. We have $\sigma \sim^* \tau'$. Let $\sigma' = \tau'$ and we are done.

Case $\tau = \tau_1 \tau_2$: Partition the free variables $\sigma$ into three lists: $\sigma_1$ are the variables appearing only in $\tau_1$, $\sigma_2$ are the variables appearing only in $\tau_2$, and $\sigma_3$ are the variables appearing in both. We remember the index of each $\sigma$ in the original $\pi$. Partition $\sigma$ into $\sigma_1, \sigma_2$, and $\sigma_3$, corresponding to the partition of $\pi$. We see that $\tau_1[\sigma_1/b_1][\sigma_3/b_3] \sim \tau_2[\sigma_2/b_2][\sigma_3/b_3] \sim \tau'$. This must be by RED_App. Thus, $\tau_1 \sim \tau_2$ and $\tau_1[\sigma_1/b_1][\sigma_3/b_3] \sim \tau_1'$ and $\tau_2[\sigma_2/b_2][\sigma_3/b_3] \sim \tau_2'$. Then we use the induction hypothesis to get $\tau_1', \tau_3'$ such that $\tau_1' \sim \tau_1[\sigma_1/b_1][\sigma_3/b_3]$. Similarly, we get $\tau_2', \tau_3'$ such that $\tau_2' \sim \tau_2[\sigma_2/b_2][\sigma_3/b_3]$.

The induction hypothesis also tells us $\sigma_1 \sim^* \sigma_1', \sigma_3 \sim^* \sigma_3', \sigma_2 \sim^* \sigma_2', \sigma_4 \sim^* \sigma_4'$. We use confluence (Lemma 15) to get $\sigma_{30}$ such that $\sigma_{31} \sim^* \sigma_{31}$ and $\sigma_{32} \sim^* \sigma_{32}$. By Lemma 14, we see that $\tau_1[\sigma'_{11}/b_1][\sigma'_{31}/b_3] \sim^* \tau_1[\sigma_1/b_1][\sigma_3/b_3]$ and similarly for $\tau_2$. Let $\sigma$ be the list of types made from $\sigma_1, \sigma_2$, and $\sigma_{30}$, undoing the partition to make the $b$ earlier. We can now conclude $\tau_1[\sigma'/\pi] \sim \tau_2[\sigma'/\pi] \sim^* \tau_2[\sigma'/\pi] \sim^* \tau_2[\sigma'/\pi]$ as desired.

Case $\tau = \pi_1 \pi_2$: Impossible, by assumption.

We need to bring in notions of flattening and apartness from Eisenberg et al. (2014):

**Definition 17** (Flattening). To flatten a type $\tau$ into $\tau'$, written $\tau' = \text{flatten}(\tau)$, process the type $\tau$ in a top-down fashion, replacing every type family application with a type variable. Two or more syntactically identical type family applications are flattened to the same variable; distinct type family applications are flattened to distinct fresh variables.

**Definition 18** (Apartness). To test for apartness $\tau_1, \tau_2$, let $\tau_1' = \text{flatten}(\tau_1)$ and $\tau_2' = \text{flatten}(\tau_2)$ and check $\text{unify}_\sigma(\tau_1', \tau_2')$. If this unification fails, then $\tau_1$ and $\tau_2$ are apart. More succinctly: $\text{apart}(\tau_1, \tau_2) = \neg\text{unify}_\sigma(\text{flatten}(\tau_1), \text{flatten}(\tau_2))$.

**Property 19** ($U$ and apartness). If $U(\tau_1, \tau_2) \neq \text{nothing}$, then apart($\tau_1, \tau_2$).
This property holds by the construction of algorithm $U$. Note that $U$ returns Nothing only when there is a direct type constant mismatch.

**Lemma 20** (Apartness is stable under substitution). If apart($\tau_1, \tau_2$), then for all substitutions $\theta$, apart($\theta(\tau_1), \theta(\tau_2)$).

**Proof.** Proved as Property 12 of Eisenberg et al. (2013). The generalization used here does not change the substance of the proof.

In order to prove the soundness of the injectivity check, we will need to know that apart types are not joinable. It is hard to prove this statement directly, as it is equivalent to proving consistency of the type system with both non-linear types and non-terminating type families. As argued in Eisenberg et al. (2014) and Breitner et al. (2014a), this proof seems to require a positive answer to RTA Open Problem #79.

**Assumption 21** (Apart types are not joinable). If apart($\tau_1, \tau_2$), then $\neg(\tau_1 \Leftarrow \tau_2)$.

Previous work requires an assumption of either linear patterns or termination to prove soundness. We are no different, and the above assumption is not, by itself enough. So, from here on out, we will assume that all type family reductions terminate.

**Assumption 22** (Termination). There exists no infinite (non-reflexive) chain of rewrites.

**Definition 23** (Algorithm $U^*$). Define Algorithm $U^*$ to be identical to Algorithm $U$ with the difference that line 7 is missing.

The removed line is the one that treats occurrences of injective type families differently than other type families. The proof proceeds by showing that an injectivity check based on $U^*$ is sound. We then add back in the special treatment of injective type families.

**Property 24** (Pre-unifier). If $U^*(\tau_1, \tau_2) \ni \sigma$, then $\exists \theta, \exists \phi$ such that $U^*(\tau_1, \tau_2) \ni \sigma \Leftarrow \phi(\tau_1)$ and $\phi(\tau_2)$.

In order to prove the soundness of the injectivity check, we will need to know that apart types are not joinable. It is hard to prove this statement directly, as it is equivalent to proving consistency of the type system with both non-linear types and non-terminating type families. As argued in Eisenberg et al. (2014) and Breitner et al. (2014a), this proof seems to require a positive answer to RTA Open Problem #79.
We thus know that the injectivity check using $U^*$ is sound. But is it sound for $U$? The output of $U$ will, perhaps, have more mappings than the output for $U^*$, as $U$ can look under injective type family applications. Thus, $U(\tau_1, \tau_2)$'s output might not be a pre-unifier of flatten($\tau_1$) and flatten($\tau_2$). But the extra mappings are harmless: they must be between two joinable types. The details of how to prove this more formally have eluded us thus far. Nevertheless, we conjecture that, since $U^*$ is sound, $U$ is too, given the straightforward nature of the difference between the two algorithms.

Conjecture 26 (Injective type families). If $F \vdash \sigma$ and $F$ is $i$-injective, then $\tau_1 \Rightarrow \sigma_1$.

We are now in a position of finishing up our proof of consistency.

Lemma 27 (Completeness of the rewrite relation). If $\emptyset \vdash \gamma : \tau_1 \Rightarrow \tau_2$, then $\tau_1 \Rightarrow \tau_2$.

Proof. By induction on the structure of $\emptyset \vdash \gamma : \tau_1 \Rightarrow \tau_2$. This follows previous work (such as Breitner et al. (2014b)), with one additional case:

Case CO_NTHYFAM:

$\Gamma \vdash \gamma : F \vdash F \sigma$

$F$ is $i$-injective

$\Gamma \vdash \text{nth}(\gamma)(\tau_1) \Rightarrow \sigma_1$, CO_NTHYFAM

By the induction hypothesis and Conjecture 26.

Theorem 28 (Consistency). The global context is consistent.

Proof. Proved as Lemma 36 in Breitner et al. (2014b).

We have one final link to make in order to connect back with Property 7. We want Lemma 25 to prove soundness, but that lemma is stated in terms of $\Rightarrow$, and the soundness property concerns $\sim$. With the following lemma, we see that these relations are one and the same, and so indeed we have addressed soundness as desired.

Lemma 29 (Soundness of the rewrite relation). If $\tau \Rightarrow \sigma$ (where both $\tau$ and $\sigma$ have no free variables), then there exists a $\gamma$ such that $\emptyset \vdash \gamma : \tau \sim \sigma$.

Proof. We prove that if $\tau \sim \tau_0$ (for closed $\tau$), then $\gamma$ exists such that $\emptyset \vdash \gamma : \tau \sim \tau_0$. We are done by straightforward induction on the structure of the proof that $\tau \sim \tau_0$. Note that all rules in the rewrite relation correspond exactly to coercion forms.

D. Completeness of the injectivity check

Property (Completeness (Property 8)). Suppose a type family $F$ has equations such that for all right-hand sides $\tau$:

- $\tau$ is type-family-free,
- $\tau$ has no repeated use of a variable, and
- $\tau$ is not a bare variable.

If $F$ is $n$-injective, then the injectivity check will conclude that $F$ is $n$-injective.

As described in Section 4.4, the conditions significantly reduce the complexity of the injectivity check. Under these conditions, the injectivity check is equivalent to the following:

Definition 30 (Restricted injectivity check). Assuming the conditions stated in Property 8, a type family $F$ is $n$-injective iff, for every pair of equations $F \sigma_i = \tau_i$ and $F \sigma_j = \tau_j$ (including $i = j$):

1. $\tau_i$ and $\tau_j$ do not unify, or
2. $\tau_i$ and $\tau_j$ unify with substitution $\theta$, and
   (a) $\theta(\sigma_i) = \theta(\sigma_j)$, or
   (b) $F \theta(\sigma_j)$ cannot reduce via equation $i$, or
   (c) $F \theta(\sigma_j)$ cannot reduce via equation $j$.

The completeness property is then directly implied by the following lemma:

Lemma 31 (Completeness). Suppose $F$ conforms to the conditions in Property 8. If, for all $\tau$ and $\sigma$, $F \tau \sim F \sigma$ implies $\tau_n \sim \sigma_n$, then the restricted injectivity check will succeed on $F$.

Proof. We prove the contrapositive, that when restricted injectivity check fails on $F$, there exist $\tau_0$ and $\sigma_0$ such that $F \tau_0 \sim F \sigma_0$ and $\tau_0 \sim \sigma_0$.

For the restricted injectivity check to fail, there must be equations $F \tau_0 = \tau_i$ and $F \sigma_0 = \tau_j$ such that $\tau_i$ and $\tau_j$ unify with substitution $\theta$. Further, it must be that $\theta(\sigma_i) \neq \theta(\sigma_j)$ and that both $F \theta(\sigma_i)$ and $F \theta(\sigma_j)$ can reduce by their respective equations.

Choose $\tau_0 = \theta(\sigma_i)$ and $\sigma_0 = \theta(\sigma_j)$. We can see that $F \tau_0$ reduces to $\theta(\tau_i)$ and $F \sigma_0$ reduces to $\theta(\tau_j)$. Because $\theta$ is a unifier for $\tau_i$ and $\tau_j$, we see that $F \tau_0 = F \sigma_0$ and by reflexivity of $\sim$, we know $F \tau_0 \sim F \sigma_0$.

We still must show $\theta(\tau_0) \neq \theta(\sigma_0)$. By our choice of $\tau_0$ and $\sigma_0$, we see that $\tau_0 = \theta(\sigma_i)$ and $\sigma_0 = \theta(\sigma_j)$. We know from our failed injectivity check, then, that $\tau_0 \neq \sigma_0$. Is it still possible that $\tau_0 \sim \sigma_0$? Though we know that $\tau_0$ and $\sigma_0$ are type-family-free, because $\sigma_i$ and $\sigma_j$ are type-family-free (because they are left-hand sides of a type family equation) and $\theta$ is type-family-free (because it is a most-general unifier of two type-family-free types). If $\tau_0 \sim \sigma_0$ were to hold, then, by completeness of the rewrite relation (Lemma 27), that $\tau_0 \sim \sigma_0$. But the type reduction relation coincides with equality when no type families are involved. Thus, if $\tau_0 \sim \sigma_0$, it must be that $\tau_0 = \sigma_0$, which is a contradiction.

Thus, we are done.